VCSMS PRIME

Program for Inducing Mathematical Excellence
Syllabus
Carl Joshua Quines

Schedule. Thursday and Friday, 3:30 PM to 5:30 PM at the math lab.

Homework. There will be 6 problem sets, due on Wednesdays except the final problem set. These will refer to last year's PRIME problems, which can be found in `http://www.cjquines.com/files/prime.pdf`. All problem sets must be submitted completely. The lowest scoring set will be dropped.

Late and incomplete policy. Late problem sets will be given partial credit. If you have an incomplete problem set, you *must* submit the problems you did not solve. Students who do not answer all problems will be marked as incomplete.

Exams. There will only be one two-hour final exam, held on session 12, as a mock qualifying stage.

Grading. Weekly homework is 50%, the final exam is 30%, and in-class participation is 20%.

Calendar. There will be twelve sessions this year, followed by the PMO Qualifying Stage. Session 11 is after exams, for two hours; session 12 is for the whole day. Grades will be given out on session 12.

1. Th, September 14: Angles and areas
2. F, September 15: Functions

W1. W, September 20: Week 1 homework due

3. Th, September 21: Combinatorial principles
4. F, September 22: Counting and probability

W2. W, September 27: Week 2 homework due

5. Th, September 28: Trigonometry
6. F, September 29: Circles and polygons

W3. W, October 4: Week 3 homework due

7. Th, October 5: Sequences and inequalities
8. F, October 6: Algebraic manipulation

W4. W, October 11: Week 4 homework due

9. Th, October 12: Coordinates
10. F, October 13: Polynomials

W5. W, October 18: Week 5 homework due

* W, October 18: GMATIC
* Th–F, October 19–20: Exams

11. F, October 20: Number theory

* M–W, October 23–November 1: Sembreak

W6. F, October 27: Week 6 homework due

12. F, October 27: Metasolving

* F, October 27: Final exam
* F, October 27: Grades
* S, October 28: PMO Qualifying Stage

Reminders

- **Some portion of your grade in Mathematics will be your grade in PRIME.** You are expected to work on homework in lieu of some of your activities in math class. In particular, *you will have enough time for homework.*
- You can discuss homework with me or your classmates any time. If you get stuck on a problem, ask.
- Remember to look at our Facebook group for updates!

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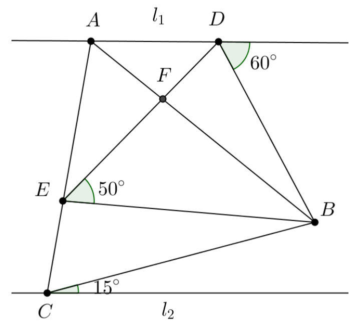
Session 1: Angles and Areas

September 14, 2017

Lecture problems



1. (QI6) Three circles with radii 4, 5, and 9 have the same center. If $x\%$ of the area of the largest circle lies between the other two circles, what is x to the nearest integer?
2. (QI12) In the figure on the right (not drawn to scale), triangle ABC is equilateral, triangle DBE is isosceles with $EB = BD$, and the lines l_1 and l_2 are parallel. What is $m\angle FBE$?
3. (QI14) A regular hexagon with area 28 is inscribed in a circle. What would the area of a square inscribed in the same circle be?
4. (AI13) A circle is inscribed in a 2 by 2 square. Four squares are placed on the corners (the spaces between circle and square), in such a way that one side of the square is tangent to the circle, and two of the vertices lie on the sides of the larger square. Find the total area of the four smaller squares.
5. Lines l_1 and l_2 are parallel. Points A and B are in l_1 and points C and D are in l_2 such that A and D are on opposite sides of line BC. Point X is in between lines l_1 and l_2 such that $\angle BAX = 37°$ and $\angle XCD = 23°$. What is $m\angle AXC$?
6. In a square, four semicircles centered on the midpoints of its sides are drawn inward. Find the total area of the regions common to two semicircles.
7. The medians to the legs of an isosceles triangle are perpendicular to each other. If the base of the triangle is 4, find its area.

Angles

- Circles make lots of angle relations, especially tangents to circles.
- Parallel lines make lots of equal angles, see problem 2.
- Don't be afraid to add additional lines, as in the sigma problem.
- Draw large diagrams. Rule: if you cannot make any more markings, draw a larger diagram.

Areas

- For a triangle ABC with side lengths a, b, c, altitudes to sides BC, CA, AB h_a, h_b, h_c, angles A, B, C, semiperimeter s, inradius r and circumradius R, its area is

$$[ABC] = \frac{ah_a}{2} = \frac{1}{2}ab\sin C = \sqrt{(s-a)(s-b)(s-c)} = rs = \frac{abc}{4R}.$$

- Similar figures with ratio k have ratio of areas k^2: look at the trapezoid.
- Break down into smaller areas and then add and subtract them, like problems 1, 3, 6.
- Length chase. As in problems 4 and 7.

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Session 2: Functions

September 15, 2017

Lecture problems

1. (QI1) If $27^3 + 27^3 + 27^3 = 27^x$, what is the value of x?
2. (AI11) How many real numbers x satisfy the equation
$$\left(\left|x^2 - 12x + 20\right|^{\log x^2}\right)^{-1+\log x} = \left|x^2 - 12x + 20\right|^{1+\log(1/x)}?$$
3. (QI2) Let $a, b > 0$. If $|x - a| \leq a + b$, what is the minimum value of x?
4. (QIII4) Let $f(x) = \sqrt{-x^2 + 20x + 400} + \sqrt{x^2 - 20x}$. How many elements in the range of f are integers?
5. (AI2) Let f be a real-valued function such that $f(x - f(y)) = f(x) - xf(y)$ for any real numbers x and y. If $f(0) = 3$, determine $f(2016) - f(2013)$.
6. (AI8) For each $x \in \mathbb{R}$, let $\{x\}$ be the fractional part of x in its decimal representation. For instance, $\{3.4\} = 3.4 - 3 = 0.4$, $\{2\} = 0$, and $\{-2.7\} = -2.7 - (-3) = 0.3$. Find the sum of all real numbers x for which $\{x\} = \dfrac{1}{5}x$.

Exponents

- $b^e = x$. If $b > 0$ (and not 1) then $e \in \mathbb{R}$. If $b = 0$, then $e > 0$. If $b = 1$, then range is just 1. The negative case is very complicated. Range is all real numbers, except $b \leq 0$ and $b = 1$. Monotonic, so if $e \in [c, d]$ then $x \in [b^c, b^d]$.
- Write everything in the same base and hope it works!
- If we can't make the bases the same, we can make the exponents the same: 11^8 and 16^7.
- If $a, b, c \in \mathbb{R}$, and $a \geq 0$ then $a^b = a^c$ implies one of either: **a)** $a = 0, b, c > 0$, **b)** $a > 0, b = c$, **c)** $a = 1$. The case of negative base is complicated again.

Logarithms

- $\log_b x = e$. Must have $b > 0$ and $x > 0$, but range is any $e \in \mathbb{R}$. Monotonic, so if $x \in [c, d]$ then $e \in [\log_b c, \log_b d]$.
- Write everything in the same base and hope it works!
- Spam $\log_b x = e \iff x = b^e$. Think of "raising both sides to the bth power" and "cancelling the logarithm:" $b^{\log_b x} = x$. Since logarithms are monotonic, inequalities work too.
- Recall the rules of logarithms: the most important are $\log_b x + \log_b y = \log_b xy$, $c \log_b x = \log_b x^c$, and $\log_b x = \frac{\log_c x}{\log_c b}$, the rest can be derived.

Surds

- $y = \sqrt{x}$. Must have $x \geq 0$. Monotonic, so if $x \in [a, b]$ then $y \in [\sqrt{a}, \sqrt{b}]$.
- Rationalize the denominator, often with $x^2 - y^2 = (x-y)(x+y)$ or $x^3 \pm y^3 = (x \pm y)(x^2 \mp xy + y^2)$.
- If you have $\sqrt{a + \sqrt{b}}$, maybe you can simplify it to $x + \sqrt{y}$. Equate and square both sides. Same thing with cube roots.
- If you have conjugates, like $x = \sqrt{a} + \sqrt{b}$ and $y = \sqrt{a} - \sqrt{b}$, you can often write y in terms of x.

Floor, ceiling, fractional

- $\lfloor x \rfloor$ is the integer part of x. If $\lfloor x \rfloor = c$, then $c \leq x < c + 1$. Monotonic.
- $\lceil x \rceil$ is ceiling, if $\lceil x \rceil = c$ then $c < x \leq c + 1$. Monotonic.
- $\{x\}$ is fractional part or $x - \lfloor x \rfloor$. *Not monotonic.* Always has $0 \leq \{x\} < 1$.
- One technique is to substitute $x = n + r$ where $n = \lfloor x \rfloor$ and $r = \{x\}$. Use the fact that $0 \leq r < 1$ to find values of n.
- Another technique is to replace all $\{x\}$ with $x - \lfloor x \rfloor$.

Absolute value

- $y = |x|$ is always split into two cases: when $x < 0$, $|x| = -x$ and when $x > 0$, $|x| = x$.
- *Not monotonic*, so we have to be careful with inequalities: if you have $|x| \leq y$ then you split it into $-y \leq x \leq y$. If $|x| \geq y$ then $x \leq -y$ or $x \geq y$.
- Sums of absolute values: if you're minimizing $|x - a| + |x - b| + |x - c|$, the minimum value is when x is the median of a, b, c. If even number of values, then any x between the two median values works.

Rational functions and limits

- $y = \frac{f(x)}{g(x)}$. Must have $g(x) \neq 0$.
- Very common to find the range, as in $\frac{x^4+3}{2x^4+1}$. Find the fastest growing term and consider that. What happens if $x \to \infty$, or $x \to -\infty$? What makes it the smallest value?
- From slow growing to fast: constants, logarithms, polynomials, exponents. (This is towards positive infinity.)

Functional equations

- Treat it as a system of equations machine and find stuff.
- Substitution: To find $f(0)$ or $f(1)$ or whatever, get stuff to cancel. Try substituting all 0 or all 1.
- Involutions: if we have $f(x)$ and $f(a - x)$ and we're finding $f(c)$, then substituting $x = c$ and $x = a - c$ gives two equations. Similar: $f(x)$ and $f(1/x)$ means substituting $x = c$ and $1/c$. Functions where $f \circ f(x) = x$ are called *involutions*.
- Induction: if we have $f(x)$ and $f(x + 1)$ and you know $f(0)$, you can find any $f(n)$ for any natural n.
- Cheat: if only one function satisfies the conditions (i.e. there's only one possible answer), then just find one and use that. Try linear functions, constants, etc.
- Cauchy FE: if $f(x + y) = f(x) + f(y)$ for $x, y \in \mathbb{Q}$ then $f(x) = kx$ for some constant k. Making it reals is harder, it works if you have either bounding, monotonicity, or continuity.

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Session 3: Combinatorial Principles

September 21, 2017

Lecture problems

1. (QII1) I have 2016 identical marbles. I plan to distribute them equally into one or more identical containers. How many ways can this be done if I have an unlimited number of containers?
2. (QI13) How many three-digit numbers have distinct digits that add up to 21?
3. (QIII2) Using the numbers $1, 2, 3, 4, 5, 6$ and 7, we can form $7! = 5040$ 7-digit numbers in which the 7 digits are all distinct. If these numbers are listed in increasing order, find the 2016th number in the list.
4. A set of points is chosen on the circumference of a circle so that the number of different triangles with all three vertices among the points is equal to the number of pentagons with all five vertices in the set. How many points are there?
5. Compute $\binom{1000}{0} + \binom{1000}{2} + \binom{1000}{4} + \cdots + \binom{1000}{1000}$.
6. (AI15) How many numbers between 1 and 2016 are divisible by exactly one of $4, 6$, or 10?
7. (QII9) How many ordered triples of positive integers (x, y, z) are there such that $x + y + z = 20$ and two of x, y, z are odd?

Counting is multiplication

- The number of ways to choose in a series of independent choices is the product of the number of choices. When it's in a row (or a circle, or whatever) it's called a *permutation*.
- Problem 1: The formula for the number of divisors is $\prod(e_i + 1)$ because for each prime p_i, a divisor can have either $p_i^0, p_i^1, p_i^2, \ldots, p_i^{e_i}$ as a factor.
- Problem 2: Sometimes you just have to count brute force. Then multiply by 3! to account for the number of arrangements, but be careful with zero. General rule: $n!$ ways to arrange n items in a row.
- Problem 3: This is like "the permutations of VALMASCI are sorted alphabetically, what is the nth word in the list?" This is called *lexicographical order*, and you count by going left-to-right, slot-by-slot. How many with first digit 1? First digit 2? Then if you go over 2016 you move to the next digit.

Counting is division

- When order matters, it's a permutation, but when it doesn't it's a combination. Sometimes we *overcount*, and we want to count things as the same. How many ways to arrange letters in PHILIPPINES?
- Permutations on a circle: divide by n. With reflections: divide by 2.
- When order doesn't matter, it's called *combination*: divide by $k!$. This gives binomial coefficients $\binom{n}{k}$.
- Problem 4: A triangle is defined by its three vertices, same for a pentagon, so $\binom{n}{3} = \binom{n}{5}$. Cancel.
- Why "binomial coefficient"? Because it's the coefficients in $(x + y)^n$.
- Problem 5: What happens when we expand $(1 + 1)^{1000}$ and $(1 - 1)^{1000}$ using the binomial theorem?

Counting is addition (and subtraction)

- Draw Venn diagrams and add and subtract. For PIE, the pattern is add, subtract, add, subtract, add, subtract.
- Problem 6: Add divisble by $4, 6, 10$, subtract twice divisible by $\mathrm{lcm}(4, 6) = 12, \mathrm{lcm}(4, 10) = 20, \mathrm{lcm}(6, 10) = 30$, add thrice divisible by $\mathrm{lcm}(4, 6, 10) = 60$.

Counting is bijection

b balls	u urns	no restriction	at most 1	at least 1
D	D			(hard)
I	D			
D	I	(hard)		(hard)
I	I	(hard)		(hard)

- Twelvefold way: distinguishable vs indistinguishable balls (2), distinguishable vs indistinguishable urns (2), no restriction vs at most one ball in each urn vs at least one ball in each urn (3).
- We know how to do five out of twelve:
  - How many ways to give three different candies to five children? (5^3)
  - How many ways to give three different candies to five children, so that each child gets at most one candy? ($5 \times 4 \times 3$)
  - How many ways to give three identical candies to five children, so that each child gets at most one candy? ($\binom{5}{3}$)
  - How many ways to place three different candies in five identical pouches, so that each pouch has at most one candy? (1)
  - How many ways to place three identical candies, in five identical pouches, so that each pouch has at most one candy? (1)
- Another five out of twelve have no simple formula, and should be done manually.
- The remaining two can be done with balls and urns arguments:
  - How many ways to give three identical candies to five children? ($\binom{3+5}{3}$)
  - How many ways to give three identical candies to two children, so each child gets at least one candy? ($\binom{3-1}{2-1}$)
- Appears in lots of ways:
  - $w + x + y + z = c$ for nonnegative or positive integers.
  - How many ways to roll 5 dice so the sum is 16?
  - Problem 7: There are $\binom{3}{2}$ ways to pick the two odd numbers. Rewrite as $2a + 1, 2b + 1, 2c$, and then do balls and urns, then remember to multiply by $\binom{3}{2}$.
  - Selecting 6 cookies out of 3 different kinds, when you have unlimited supplies of each kind.
  - Number of 7-digit numbers with digits in increasing order.
  - How many ways to arrange 3 maple trees, 4 oak trees and 5 birch trees in a row so no two birch trees are adjacent?

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Session 4: Counting and Probability

September 22, 2017

Lecture problems

1. (QI7) Issa has an urn containing only red and blue marbles. She selects a number of marbles from the urn at random and without replacement. She needs to draw at least N marbles in order to be sure that she has at least two red marbles. In contrast, she needs three times as much in order to be sure that she has at least two blue marbles. How many marbles are there in the urn?
2. I flip a coin. If it comes up heads I roll two dice and if it comes up tails I roll three dice. Given that the sum of the dice is 4, what is the probability I flipped heads?
3. At a nursery, 2006 babies sit in a circle. Suddenly, each baby randomly pokes either the baby to its left or its right. What is the expected number of unpoked babies?
4. (QII7) Louie plays a game where he throws a circular coin with radius 1 unit, which falls flat entirely inside a square board having side 10 units. He wins the game if the coin touches the boundary or the interior of a circle of radius 2 units drawn at the center of the board. What is the probability that Louie wins the game?
5. (QII8) Guido and David each randomly choose an integer from 1 to 100. What is the probability that neither is the square of the other?
6. (AI7) A small class of nine boys are to change their seating arrangement by drawing their new seat numbers from a box. After the seat change, what is the probability that there is only one pair of boys who have switched seats with each other and only three boys who have unchanged seats?
7. (AI18) A railway passes through four towns A, B, C, and D. The railway forms a complete loop, as shown on the right, and trains go in both directions. Suppose that a trip between two adjacent towns costs one ticket. Using exactly eight tickets, how many distinct ways are there of traveling from town A and ending at town A? (*Note that passing through A somewhere in the middle of the trip is allowed.*)

Shooting pigeons

- Existence: is there or is there not? Versus enumerative: how many?
- Pigeonhole principle: if you have n pigeons and $n+1$ holes then at least one pigeon will have more than one hole. (Or if you put $n+1$ pigeons into n holes, but that's more boring.)
- Drawer principle: if you have a red socks and b non-red socks in a drawer, you have to pick $b+1$ socks to assure you get at least one red sock.
- Problem 1: direct application of drawer principle.
- Think of "worst-case" scenario and go from there. Related:
  - How many socks to pick to ensure at least one of each color?
  - How many integers do you have to pick to ensure some two of these integers have a difference that is divisible by four?
  - How many points do you have to pick in a square of side 2 to ensure two of them are within distance $\sqrt{2}$ of each other?

Probabilities

- Problem 2: Conditional probability: $P(A|B) = P(AB)/P(B)$.
- Expected value is its "average". Expected value of a dice roll is 3.5. Expected number of heads in three coin tosses is 1.5. It is linear: if you roll 100 dice you expect an average of 350. But this is true even if it is not independent.
- Problem 3: Just consider one baby and multiply by 2006.
- Geometric probability is dividing the areas. Sometimes it comes up in unexpected ways, like when choosing real numbers (dividing lengths). Sometimes it is helpful to graph and find the areas.
- Problem 4: Draw a picture! Key idea is to consider center of coin: where can it fall so that Louie wins?

Counting the wrong thing

- Sometimes it is easier to count the opposite and subtract: *complementary counting*.
- How many ways to sit six people in a row so two particular people don't sit next to each other?
- I choose four letters at random, what's the probability I chose the same letter twice?
- Problem 5: Instead of "neither is the square of the other", what about the probability one of them is the square of the other?

Brute force

- Count manually. Five cases out of twelvefold way.
- How many ways to rearrange four letters so no letter is in its own place and no two letters have swapped?
- Problem 6: There are $\binom{9}{2}$ ways to pick two boys who switch, and $\binom{7}{3}$, out of the remaining 7 boys, to pick three unchanged. Then there are 6 ways for the remaining four boys.
- Divide into cases: how many odd numbers with middle digit 5, no digit repeated, between 20000 and 69999?

"Carefully"

- How do you solve hard combinatorics problems?
- Problem 7: How can represent a trip? We can do it using towns, but that would be too hard. It is easier to represent with "left" and "right". Divide into cases and use the identity from yesterday.
- Often problems don't involve only one technique, but several. Make sure to remember all the principles, be creative, and try many different ways until you find one that works.

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Session 5: Trigonometry
September 28, 2017

Lecture problems

1. (AIME 1995/7) Given $(1+\sin t)(1+\cos t) = \frac{5}{4}$, find $(1-\sin t)(1-\cos t)$.
2. (QI9) Evaluate the following sum: $1+\cos\frac{\pi}{3}+\cos\frac{2\pi}{3}+\cos\frac{3\pi}{3}+\cdots+\cos\frac{2016\pi}{3}$.
3. (Huang[1]) Divide $\sin 3x\,(2\cos 2x - 1)$ by $\sin x\,(2\cos 4x+1)$ and simplify.
4. (AIME 1996/10) Find the smallest positive integer solution to $\tan 19x^\circ = \dfrac{\cos 96^\circ + \sin 96^\circ}{\cos 96^\circ - \sin 96^\circ}$.
5. (AI6) Find the exact value of $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)$.
6. (Morrie's Law) Simplify $\cos 20^\circ \cos 40^\circ \cos 80^\circ$.
7. Find the exact value of $\cos\pi/5$.
8. (PEM 2016/10) Find the ratio of $\sin 1^\circ + \sin 2^\circ + \cdots + \sin 44^\circ$ to $\cos 1^\circ + \cos 2^\circ + \cdots + \cos 44^\circ$.
9. (Stewart's Theorem) In triangle ABC, point D is on line BC. Let $AD = d, BD = m$ and $CD = n$. Then $man + dad = bmb + cnc$.

As ratios of sides

- In terms of triangles: sine, cosine, tangent, secant, cosecant, cotangent, are ratios of sides. Example: find $\tan\alpha$ given $\sin\alpha = \frac{1}{2}$. This is the classical development in classroom math.
- Values for special angles: $30^\circ, 45^\circ, 60^\circ$ can be derived from special right triangles $30^\circ - 60^\circ - 90^\circ$ (equilateral triangle), and $45^\circ - 45^\circ - 90^\circ$ (square). We can get 0° and 90° from reasoning.
- Identities: cofunctions, Pythagorean identities.
- Problem 1: Abuse symmetry: replace sums and products with variables. Pythagorean identity.

As lengths in a circle

- In terms of circles: the functions are lengths in the unit circle: $(\sin\theta, \cos\theta)$. Tangent is the length of the tangent to x-axis, cotangent is length to y-axis. Secant and cosecant are x and y intercepts.
- Problem 2: Radians are more natural than degrees: 2π radians is 360°.
- Identities: reflection over $0, \frac{\pi}{4}, \frac{\pi}{2}$, shifts by $\frac{\pi}{2}, \pi, 2\pi$, Pythagorean identities, $\csc^2\theta + \sec^2\theta = (\cot\theta + \tan\theta)^2$.

As complex numbers

- Euler's identity says $e^{ix} = \cos x + i\sin x$. Cosine is the real part and sine is the imaginary part.
- If $z = e^{ix}$ then $1/z = e^{-ix}$. We can state $\cos x$ and $\sin x$ in terms of z only. Also, by de Moivre, $\left(e^{ix}\right)^n = \cos nx + i\sin nx$. We can find $\cos nx$ and $\sin nx$ in terms of z only.
- Identities: reflection, Pythagorean identities.
- Problem 3: Let $z = e^{ix}$. Substitute sine and cosine. Everything cancels nicely.

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[1]From Complex Numbers in Trigonometry, `https://aops.com/community/c6h609795`. Read it, it's good.

Sum and difference

- Fundamental is $\sin(x+y)$. Many derivations: stick two right triangles with angles x, y and common side and find the area; use the unit circle and rotate; Euler's identity.
- Derive $\sin(x \pm y), \cos(x \pm y), \tan(x \pm y), \cot(x \pm y)$. Triple tangent formula: $x + y + z = \pi$ iff $\tan x + \tan y + \tan z = \tan x \tan y \tan z$.
- Problem 4: Inspired by tangent sum formula. Force tangent by dividing by $\cos 96°$ and $\tan 45° = 1$.
- Inverses for trigonometric functions exist. Derive $\tan^{-1} x \pm \tan^{-1} y$, $\cot^{-1} x \pm \cot^{-1} y$ by subtituting in the sum and difference formulas.
- Problem 5: The shortcut for $\tan^{-1}(p_1/q_1) + \tan^{-1}(p_2/q_2)$ is "cross multiply and add for numerator, product of denominators minus product of numerators for denominator."
- Derive $\sin 2x$. Write $\cos 2x$ in three ways using Pythagorean identity, and use that to derive half-angle formulas. (Sine is minus, same as in complex numbers.) There is also derivation with Euler's formula.
- Problem 6: The doubling inspires us to force double angle formula. Let expression be x and multiply both sides by $2\sin 20°$. (Interestingly, $\tan 20° \tan 40° \tan 80° = \tan 60°$.)
- Problem 7: This is important. Let $a = \cos \pi/5$ and $b = \cos 2\pi/5$. Then use double angle formulas on a and b, but $\cos 4\pi/5 = -\cos \pi/5 = -a$.

Prosthaphaeresis

- Greek *prosthesis* means addition, *aphaeresis* means subtraction. Which is how you derive them: $\cos x \cos y, \sin x \sin y, \sin x \cos y$ by cancelling out $\cos(x+y)$ and $\cos(x-y)$, etc.
- Reverse formulas: find $\sin x \pm \sin y$ by reversing the prosthaphaeresis formulas. As in, let $x' = x - y$ and let $y' = x + y$ and rewrite.
- Problem 8: Looks like arithmetic sequence. We did arithmetic sequence by pairing up opposite terms. Here, we pair $\sin 1°$ and $\sin 44°$ and use sum-to-product, etc. Everything cancels.

Laws

- Extended law of sines: draw the circumradius and use the definition of sine. Law of cosines: drop an altitude use the Pythagorean theorem twice.
- Problem 9: Apply law of cosines twice: on $\triangle ABD$ and $\triangle BCD$.

As functions

- Sine and cosine: domain is $\mathbb{R}$, range is $\{-1, 1\}$. Period 2π. Graphs are translations of each other.
- Cosecant and secant graphs are like a bunch of parabolas. Cosecant domain is $\mathbb{R} - \{n\pi, n \in \mathbb{Z}\}$ and secant domain is $\mathbb{R} - \{(2n+1)\pi/2, n \in \mathbb{Z}\}$. Range is $(-\infty, -1] \cup [1, \infty)$.
- Tangent and cotangent: tangent domain is $\mathbb{R} - \{(2n+1)\pi/2, n \in \mathbb{Z}\}$ and cotangent domain is $\mathbb{R} - \{n\pi, n \in \mathbb{Z}\}$. Range is $\mathbb{R}$.
- Inverse functions: arcsin and arccos are domain $[-1, 1]$. Arcsin has range $\{-\pi/2, \pi/2\}$, arccos has range $\{0, \pi\}$. Arctan has domain $\mathbb{R}$ and range $\{-\pi/2, \pi/2\}$: it maps the whole real line onto a finite interval.
- This means we need to be careful in solving equations like $\sin x = 1/2$, because there are infinitely many solutions. Also, $\sin^{-1} x + \cos^{-1} x = \pi/2$ and stuff like $\sin(\cos^{-1} x) = \sqrt{1 - x^2}$.

deg	rad	sin	cos	tan
0°	0			
15°	$\frac{\pi}{12}$			
18°	$\frac{\pi}{10}$			
22.5°	$\frac{\pi}{8}$			
30°	$\frac{\pi}{6}$			

deg	rad	sin	cos	tan
36°	$\frac{\pi}{5}$			
45°	$\frac{\pi}{4}$			
60°	$\frac{\pi}{3}$			
90°	$\frac{\pi}{2}$			
180°	π			

Pythagorean identity: $\sin^2\theta + \cos^2\theta =$ ______

Dividing by $\sin^2\theta$ gives: ____________________

Dividing by $\cos^2\theta$ gives: ____________________

Sum and difference:

$\sin(x \pm y) =$ ______________________________

$\cos(x \pm y) =$ ______________________________

$\tan(x \pm y) =$ ______________________________

$\cot(x \pm y) =$ ______________________________

Arctans:

$\tan^{-1} x + \tan^{-1} y =$ _________________________

$\cot^{-1} x + \cot^{-1} y =$ _________________________

$\tan^{-1}\frac{p_1}{q_1} + \tan^{-1}\frac{p_2}{q_2} =$ _______________________

Half-angle:

$\sin\frac{x}{2} =$ ______________________________

$\cos\frac{x}{2} =$ ______________________________

Double angle:

$\sin 2x =$ ______________________________

$\cos 2x =$ ______________________________

$\tan 2x =$ ______________________________

$\cot 2x =$ ______________________________

Product-to-sum:

$2\cos x\cos y =$ ___________________________

$2\sin x\sin y =$ ___________________________

$2\sin x\cos y =$ ___________________________

Sum-to-product:

$\sin x \pm \sin y =$ ___________________________

$\cos x + \cos y =$ ___________________________

$\cos x - \cos y =$ ___________________________

Law of sines: ___________________________

Law of cosines: _________________________.

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Session 6: Circles and Polygons

September 29, 2017

Lecture problems

1. (AI3) Let O be the circumcenter of acute triangle ACD. The tangent to the circumcircle at A intersects line CO at B. If $AB||CD$, $AB = 6$ and $BC = 12$, what is the length of CD?
2. (QI5) The sides of a triangle are 3, 5 and x. The sides of another triangle are 4, 6 and y. If the sides of both triangles are integers, what is the maximum value of $|x - y|$?
3. Two poles of heights x and y are perpendicular to the ground. A rope is tied from the top of the first to the bottom of the second, and another rope is tied from the top of the second to the bottom of the first. The two ropes intersect at a point that is distance z from the ground. Show $1/x + 1/y = 1/z$.
4. A square is inscribed in triangle ABC such that two vertices lie on side BC, one vertex lies on AB, and the remaining vertex lies on AC. If $AB = 6$, $BC = 7$, and $CA = 8$, find the area of the square.
5. A point P and a rectangle $ABCD$ satisfy $AP = 7$, $BP = 5$, and $CP = 8$. Find DP.
6. (QI3) One diagonal of a rhombus is three times as long as the other. If the rhombus has an area of 54 square meters, what is its perimeter?
7. (AI5) In parallelogram $ABCD$, $AB = 1$, $BC = 4$, and $\angle ABC = 60°$. Supose that AC is extended from A to a point E beyond C so that ADE has the same area as the parallelogram. Find the length of DE.
8. The diagonals in trapezoid $ABCD$ with $AB||CD$ intersect at point E. Given that $[ABE] = 25$ and $[CDE] = 64$, find $[ABCD]$.
9. A quadrilateral circumscribed about a circle has side lengths $1, 2, 3, x$ in order. Find x.
10. Two regular pentagons $WORLD$ and $LAYER$ are constructed outside each other. Find the length of RL given $[ODAE] = \sqrt{5}$.

Circles

- The perpendicular bisector of a chord is a diameter. Power of a point: If chords AB and CD intersect internally at P (if P lies outside the circle it must be closer to A than B, and to C than D), then $AP \cdot BP = CP \cdot DP$.
- Problem 1: Let AO intersect CD at E. Then $\triangle COE \sim \triangle BOA$, use power of a point.
- We can find lengths of common external and internal tangents given distance of centers and radii.

Triangles

- Problem 2: By triangle inequality, $2 < x < 8$ and $2 < y < 10$. Largest difference when $x = 3, y = 9$.
- All you have to do is find similar triangles! Example: altitude of a right triangle in terms of a, b, c.
- Problem 3: Look at the bases. Add up ratios: $z/x + z/y = 1$. This comes up a lot.

Squares

- Problem 4: Divide the area of the triangle into the area above the square, the area of the square, and the area to the left and right of the square. You get side length of square as $bh/(b + h)$.

Rectangles

- A parallelogram with congruent diagonals, or one right angle.
- Problem 5: British Flag Theorem states that $AP^2 + CP^2 = BP^2 + DP^2$.

Rhombi

- A quadrilateral with all sides the same length. Its diagonals meet at right angles.
- Problem 6: Let diagonals be $6x$ and $2x$, its area is $6x \cdot 2x/2$. Then by Pythagorean, a side is $\sqrt{(3x)^2 + x^2}$.

Parallelograms

- A quadrilateral whose opposite sides are parallel. Also, a quadrilateral whose diagonals bisect each other. Also, a quadrilateral whose pairs of opposite sides are the same length. Lots of congruent triangles.
- Parallelogram law: sum of the squares of the sides is equal to sum of the squares of the diagonals.
- Problem 7: The area of ADE is twice the area of ACD, but they both have the same height, so AE is twice AC. Use law of cosines and Stewart's theorem.

Trapezoids

- A quadrilateral with *at least* (sometimes exactly) one pair of opposite sides.
- If $AB||CD$ and AD, BC intersect at E, then $\triangle ABE \sim \triangle DCE$ and $[AED] = [BEC]$. The segment connecting the midpoints of AC and BD passes through the midpoints of the diagonals and has length $(AB + CD)/2$. The segment connecting midpoints of diagonals has length $|AB - CD|/2$.
- A trapezoid is isosceles iff it is cyclic. If you have a cyclic quadrilateral with congruent opposite sides, then it's automatically a trapezoid.
- Problem 8: From $[ABE][CDE] = [AED][BEC]$ (which is true for any quadrilateral) and $[AED] = [BEC]$ we can get the answer. Generally $\sqrt{[ABCD]} = \sqrt{[ABE]} + \sqrt{[CDE]}$ for any trapezoid.

Quadrilaterals

- The quadrilateral joining the midpoints of the sides is a parallelogram, and has half the area.
- The diagonals AC and BD are perpendicular iff $AB^2 + CD^2 = BC^2 + DA^2$. Then $[ABCD] = AC \cdot BD/2$.
- If cyclic, area is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$. This is Brahmagupta's formula.

Tangential polygons

- Use the fact that the tangents to the same circle from the same point are the same length. Example: tangent lengths to the incircle. Another: can a pentagon with side lengths $2, 3, 4, 5, 6$ be tangential?
- Problem 9: By Pitot's theorem, $1 + 3 = x + 2$, so $x = 2$.

Polygons

- Polygon inequality: any side of a polygon has to be less than the sum of the other sides. If this is satisfied, you can imagine it as a wireframe and can shift it to force a lot of things.
- Don't forget MMC: how many diagonals in an n-gon? Angle of regular n-gon?
- Problem 10: I told you the trigonometric functions of $36°$ were important. Here: $\cos 36° = \frac{1}{4}\left(\sqrt{5}+1\right)$ and $\cos 72° = \frac{1}{4}\left(\sqrt{5}-1\right)$. Use $\cos a = -\cos(180° - a)$. Now law of cosines and bash.
- Whoops need to talk about three-dimensional geometry and $v - e + f = 2$.

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Session 7: Sequences and Inequalities

October 5, 2017

Lecture problems

1. (QI10) An infinite geometric series has first term 7 and sum between 8 and 9, inclusive. Find the sum of the smallest and largest possible values of its common ratio.
2. (AI4) Suppose that S_k is the sum of the first k terms of an arithmetic sequence with common difference 3. If the value of S_{3n}/S_n does not depend on n, what is the 100th term of the sequence?
3. (QII6) What is the 100th digit of the following sequence: 1 4 9 16 25 36 49 64 81 100 $\ldots$
4. (QI15) A positive integer n is a *triangular number* if there exists some positive integer k for which it is the sum of the first k positive integers, that is $n = 1 + 2 + \cdots + (k-1) + k$. How many triangular numbers are there which are less than 2016?
5. (8N4) Let $f(x) = \dfrac{2008^{2x}}{2008 + 2008^{2x}}$. Find $f(1/2007) + f(2/2007) + \cdots + f(2006/2007)$.
6. (Mock AIME 2 2007/5) Find all complex z such that $iz^2 = 1 + 2/z + 3/z^2 + 4/z^3 + \cdots$.
7. (Stanford 2011) Evaluate $\sum_{n\geq 1} \dfrac{7n+32}{n(n+2)} \cdot \left(\dfrac{3}{4}\right)^n$.
8. Find the minimum value of $x^4 + 4x + 4$.
9. (QII4) If $b > 1$, find the minimum value of $\dfrac{9b^2 - 18b + 13}{b-1}$.
10. Find the maximum value of $(1-x)(2-y)^2(x+y)$, if $x < 1$, $y < 2$ and $x + y > 0$.

Arithmetic and geometric

- You should be familiar with these. For arithmetic, $a_n = a_1 + (n-1)d$, with sum $S_n = \frac{n}{2}(a_1 + a_n)$. For geometric, $a_n = a_1 r^{n-1}$ with sum $S_n = a_1(r^n - 1)/(r-1)$. The limit as $n \to \infty$ for $|r| < 1$ gives the sum of an infinite geometric series $a/(r-1)$.
- Problem 1: If the ratio is r, then $8 \leq 7/(r-1) \leq 9$. Reciprocal function is monotonic decreasing over positive reals, so valid to say $1/9 \leq (r-1)/7 \leq 1/8$.
- Problem 2: Abuse degrees of freedom: $n = 1$ and $n = 3$ to make $S_3^2 = S_1 S_9$. Quadratic term cancels.

Digits and sequences

- Common theme: "find the xth digit after the decimal point", or "find the xth digit in the following sequence". Technique is to separate: when is the sequence one-digit, two-digit, etc. For "find the sum of the first x digits", we usually separate sum of ones-digits, tens-digits, etc.
- Problem 3: There are three one-digit squares. Then there are six two-digit squares. Squares from 10^2 to 31^2 are three-digits. There are nineteen digits more: $32^2, 33^2, 34^2$ and 35^2 each have four digits, so it's the third digit of 36^2, which is 9.

Sequence hacking

- If the dth difference is constant, then the sequence formula is a polynomial of degree d.
- Problem 4: Pretend we didn't know the formula for $0, 1, 3, 6, 10, \ldots$. The first differences are $1, 2, 3, 4, \ldots$ and the second differences are $1, 1, 1, \ldots$. This is constant, so we let $a_x = ax^2 + bx + c$ and suppose $a_0 = 0$, $a_1 = 1$, and $a_2 = 3$ to find the terms.
- Newton interpolation: Let Δ^k be the kth difference starting at zero. Then the polynomial through $(0, a_0), \ldots, (n, a_n)$ is $\sum_{k\geq 0} \binom{x}{k}\Delta^k$. For example, for $1, 3, 8, 16, \ldots$, the zeroth difference is 1, the first difference is $3-1=2$, and the second difference is $(8-3)-(3-1)=3$. So the interpolating polynomial is $\binom{x}{0}\cdot 1 + \binom{x}{1}\cdot 2 + \binom{x}{2}\cdot 3$.

Abusing symmetry

- Symmetry can come as pairing up first and last terms, or manipulating to shift the terms to the right. Sometimes we pair up subsets and their complements. Also in arithmetic and geometric sequences: we usually write in terms of middle term to become nicer.
- Problem 5: Pair up first and last terms, they have constant sum.
- Problem 6: Multiply by z to shift the terms to the right. This is an arithmetico-geometric sequence.

Forcing telescopes

- Whenever we see a polynomial denominator in a sum, we should decompose to partial fractions. Factor the denominator and then rewrite as sum of fractions with the denominators as factors.
- Problem 7: After partial fraction decomposition, we use $9 = (3/4)^2 \cdot 16$ to get stuff to telescope.

Basic inequalities

- The only real inequality is the trivial one: $x^2 \geq 0$, equality iff $x = 0$. For quadratics, we write in vertex form: $(x-h)^2 + k$, and the quadratic is thus always greater than k. Works also for, say, $x^2 + x - 12 > 0$ (or < 0), but factoring is nicer.
- Problem 8: This is $(x^2-1)^2 + 2(x-1)^2 + 2$, so its minimum value is 2 when $x = -1$.
- Separate variables if you can: if you have $2x + 3y = 1$ then you can state y in terms of x. If you have to find the minimum of $a^6 + b^4 - a^3 - b + 1$, separate to finding the minimum of $a^6 - a^3$ and $b^4 - b$.

AM-GM is life

- AM–GM is the single most important inequality in PMO. If you have a sum and you are finding the minimum, find a way to make the denominator cancel out so the right-hand-side is constant. Same thing for numerator.
- Problem 9: We want to cancel out the denominator. We long divide to get $9(b-1) + 12/(b-1)$. By AM–GM this is at least $12\sqrt{2}$.
- Problem 10: To get the product we want, we do $(1-x) + (2-y) + (2-y) + (x+y)$ for the LHS. Equality is when $1 - x = 2 - y = x + y$ or $x = 0, y = 1$.

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Session 8: Algebraic Manipulation

October 6, 2017

Lecture problems

1. (AI14) Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (2x - y, x + 2y)$. Let $f^0(x, y) = (x, y)$ and, for each $n \in \mathbb{N}$, $f^n(x, y) = f(f^{n-1}(x, y))$. Determine the distance between $f^{2016}(4/5, 3/5)$ and the origin.
2. Solve the equation $x^4 + (x - 2)^4 + 16 = 0$.
3. Factorize $x^3 + y^3 - 3xy + 1$.
4. (Sipnayan 2016) Find all x satisfying $\sqrt[3]{20x + \sqrt[3]{20x + \sqrt[3]{20x + 16}}} = 16$.
5. (AIME 1990/4) Solve $\dfrac{1}{x^2 - 10x - 29} + \dfrac{1}{x^2 - 10x - 45} - \dfrac{2}{x^2 - 10x - 69} = 0$.
6. Factorize $(x + y - 2z)^3 + (y + z - 2x)^3 + (z + x - 2y)^3$.
7. (ARML 2016) Factorize $13^4 + 16^5 - 172^2$, given it is the product of three distinct primes.
8. (AI9) Find the integer which is closest to the value of $\left(\sqrt[6]{5^6 + 1} - \sqrt[6]{5^6 - 1}\right)^{-1}$.
9. Given $x^2 - 3x + 1 = 0$, find the value of $x^5 + x^{-5}$.
10. Solve the equation $x^4 - 6x^3 - 11x^2 - 6x + 1 = 0$.
11. Prove that the product of four consecutive integers plus one is always a perfect square.
12. (MMC) Given $6x^2 + 47x + 77 = (2x + 11)(3x + 7)$, factorize 64,777.
13. Suppose $x + 3y = 3$, $y + 3z = 4$, and $z + 3x = 5$. Find x.
14. (AII1) Let x and y satisfy $\dfrac{x}{x^2y^2 - 1} - \dfrac{1}{x} = 4$ and $\dfrac{x^2y}{x^2y^2 - 1} + y = 2$. Find all possible values of xy.
15. (AIME 1989/8) Find $16x_1 + 25x_2 + 36x_3 + 49x_4 + 64x_5 + 81x_6 + 100x_7$ given that

$$\begin{aligned} x_1 + 4x_2 + 9x_3 + 16x_4 + 25x_5 + 36x_6 + 49x_7 &= 1 \\ 4x_1 + 9x_2 + 16x_3 + 25x_4 + 36x_5 + 49x_6 + 64x_7 &= 12 \\ 9x_1 + 16x_2 + 25x_3 + 36x_4 + 49x_5 + 64x_6 + 81x_7 &= 123 \end{aligned}$$

16. (AIME 1990/15) Let $f(n) = ax^n + by^n$. Given $f(1) = 3, f(2) = 7, f(3) = 16, f(4) = 42$, find $f(5)$.
17. (AIME 2014/14) Find all real solutions to $\dfrac{3}{x - 3} + \dfrac{5}{x - 5} + \dfrac{17}{x - 17} + \dfrac{19}{x - 19} = x^2 - 11x - 4$.

Abusing symmetry again

- Algebraic manipulation – substitution, factorization, manipulation – usually only has one end goal. To create symmetry. (Almost) all of the problems today deal with this.

Substitution

- Problem 1: Substituting $x, y \to 2x - y, x + 2y$ directly to $\sqrt{x^2 + y^2}$ shows that it's multiplied by $\sqrt{5}$ each time. The symmetry arises when the terms in $(2x - y)^2$ and $(x + 2y)^2$ cancel.
- Problem 2: We force symmetry by substituting $x \to y + 1$. The terms in $(y + 1)^2$ and $(y - 1)^2$ cancel.
- Problem 3: Exchanging any two variables keeps the expression the same, which means it's *symmetric*. When dealing with symmetric expressions, we usually try substituting $x + y, xy \to a, b$, because of the Fundamental Theorem of Symmetric Polynomials. This is $a^3 - 3ab - 3b + 1$, and there's an $a + 1$ factor.
- Problem 4: Substitution again. You substitute the whole expression into the 16 in the innermost infinitely many times, to get $\sqrt[3]{20x + \sqrt[3]{20x + \cdots}} = 16$, which is more symmetric. Then substitute 16 for the whole expression to get $\sqrt[3]{20x + 16} = 16$.
- Problem 5: By letting $x^2 - 10x - 29 \to a$, and *then* multiplying out, it's much easier.

Factorization

- Problem 6: There are a few factorizations you are expected to know, and one of them is $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$. But when $a + b + c = 0$, this becomes $a^3 + b^3 + c^3 = 3abc$.
- Problem 7: Use Sophie–Germain: $a^4 + 4b^4 = (a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2)$. 16^5 is 2^{20} and $13^4 - 172^2$ is $1 - 2^{10}$ via difference of two squares. Multiply both sides by $2^{10} + 1$ and then factor using Sophie–Germain on $1^4 + 4 \cdot \left(2^7\right)^4$.
- Problem 8: Factor $2 = (5^6 + 1) - (5^6 - 1)$ using difference of two squares, then two cubes.

Manipulation

- Problem 9: Divide both sides by x to get $x + 1/x = 3$. There are several ways to get $x^5 + x^{-5}$. We can take the fifth power and use the smaller powers, or we can recurse more generally by relating $x^n + x^{-n}$ and $x^{n+1} + x^{-(n+1)}$.
- Problem 10: Divide both sides by x^2 and use the substitution $x + 1/x \to a$.
- Problem 11: Suppose the smallest was n. Then $n(n + 1)(n + 2)(n + 3) + 1$. To make it easier, pair multiply $n(n + 3)$ and $(n + 1)(n + 2)$, then substitute $n^2 + 3n + 2 \to a$.
- Problem 12: Substitute $x = 100$.
- Problem 13: Add all the equations and divide by 5 to find $x + y + z = 12/5$. Subtract from the second to eliminate y and use the third to find x.
- Problem 14: Force symmetry, multiply the first by xy and subtract from the second. Solve for y in terms of x, substitute, simplify.
- Problem 15: Remember the method of finite differences? The perfect squares are quadratic, so the second difference is constant.
- Problem 16: We want to relate $f(n)$ and $f(n + 1)$. Like we did in Problem 9 for $x^n + x^{-n}$ and $x^{n+1} + x^{-(n+1)}$, we can see $f(n)(x + y) = f(n + 1) + xyf(n - 1)$.
- Problem 17: The key idea is to add 4 to both sides. The fraction $3/(x - 3)$ becomes $x/(x - 3)$, etc. Cancel out x and substitue $x \to y + 11$ for symmetry. Add opposite terms and cancel out y.

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Session 9: Coordinates

October 12, 2017

Lecture problems

1. In triangle ABC, D is the midpoint of BC, E is on segment AC such that $AE : EC = 1 : 2$, and AD and BE intersect at G. Line CG meets AB at F. Find the ratio $CG : GF$.
2. In triangle ABC, D is the midpoint of BC, E and F are on AC and AB respectively such that $AE : EC = 1 : 3$ and $AF : FB = 1 : 3$. Let DE and CF intersect at G. Compute $CG : GF$.
3. (AI19) The lengths of the two legs of a right triangle are in the ratio of $7 : 24$. The distance between its incenter and its circumcenter is 1. Find its area.
4. (AI1) The vertices of a triangle are at the points $(0, 0)$, (a, b) and $(2016 - 2a, 0)$, where $a > 0$. If (a, b) is on the line $y = 4x$, find the value(s) of a that maximizes the triangle's area.
5. (AI10) A line intersects the y-axis, the line $y = 2x + 2$, and the x-axis at the points A, B, and C, respectively. If segment AC has a length of $4\sqrt{2}$ units and B lies in the first quadrant and is the midpoint of segment AC, find the equation of the line in slope-intercept form.
6. What is the length of the shortest path from $(2, 4)$ to $(6, 2)$ that touches both the x- and y-axes?
7. (HMMT 2014) Let ABC be an acute triangle with circumcenter O such that $AB = 4, AC = 5$ and $BC = 6$. Let D be the foot of the altitude from A to BC and let E be the intersection of AO and BC. Suppose that X is on BC between D and E such that there is a point Y on AD satisfying $XY || AO$ and $YO \perp AX$. Determine the length of BX.
8. (QIII3) Let G be the set of ordered pairs (x, y) such that (x, y) is the midpoint of $(-3, 2)$ and some point on $(x + 3)^2 + (y - 1)^2 = 4$. What is the largest possible distance between any two points in G?
9. A line with slope 1 is drawn through the focus of the parabola $x^2 - 2y + 1 = 0$, intersecting its directrix at a point X. What is the sum of the slopes of the two tangent lines from X to the parabola?
10. Let $A(2, -1)$, $B(5, -3)$, and C be a point on $y = x^2$. What is the maximum value of $BC - AC$?

Mass points

- Ceva's and Menelaus' are simple consequences of mass point geometry. There are two treatments: the classical treatment, and the vectorial treatment. We'll do both, but the important idea is that the masses on both sides balance.
- Problem 1: Mass 2 to A, 1 to B and 1 to C works, so G has a mass of 4. The ratio is $3 : 1$. Alternatively, $D = \frac{1}{2}B + \frac{1}{2}C$ and $E = \frac{2}{3}A + \frac{1}{3}C$. Point G lies on both AD and BE so $G = \frac{1}{2}A + \frac{1}{2}D = \frac{1}{4}B + \frac{3}{4}E$. Subtracting C gives $G - \frac{1}{4}C = \frac{3}{4}F$ by balancing masses.
- Problem 2: We use the vectorial treatment. We have $D = \frac{1}{2}B + \frac{1}{2}C$, $E = \frac{3}{4}A + \frac{1}{4}C$, $F = \frac{3}{4}A + \frac{1}{4}B$. Then $G = kD + (1 - k)E = mC + (1 - m)F$, and we can solve: A gives $m = k$ and B gives $k = \frac{1}{3}$, which shows $G = \frac{1}{3}C + \frac{2}{3}F$, thus $CG : GF = 2 : 1$.
- Problem 3: Let $AB = 7, BC = 24, CA = 25$. We can find the lengths BD, AD, AI, CI using classical mass points and the angle bisector theorem: mass 24 on A, mass 25 on B, mass 7 on C. Then IO is a median in AIC, use Stewart's.

Cartesian plane

- Manipulating lines and slopes should be second-nature by now. Shoelace formula, distance between two points, and distance from a point to a line are important. We can treat points as vectors and use ratios.
- Problem 4: (a, b) on $y = 4x$ means $b = 4a$. Shoelace formula and maximize a quadratic.
- Problem 5: B is the midpoint of the hypotenuse of a right triangle, so BO must have length $2\sqrt{2}$ units. Set $B(x, 2x + 2)$ and use distance formula, then reflect.
- Problem 6: Reflection again! Reflect $(2, 4)$ about the y-axis and $(6, 2)$ about the x-axis, then what are the distances needed?
- Problem 7: Let D be the origin. Scale by four to eliminate fractions. Find circumradius to find distance from O to BC to find O, which is above the midpoint of BC. Find E. Set $X = rA$ and $Y = rO$ (ratios are important!) and use the perpendicular condition, which is just slopes.

Conics

- Conics are a way of thinking: each one is a locus. The parabola is a locus of points equidistant to the focus and the directrix, the ellipse the locus of points with constant sum of distances to foci, the hyperbola the locus of points with constant difference of distances to foci.
- The focus of $y = ax^2$ is $\left(0, \dfrac{1}{4a}\right)$ and its directrix is $y = -\dfrac{1}{4a}$. Just translate and rotate. Its range is from its vertex upward.
- Problem 8: Set G is a circle with half the radius.
- Problem 9: The focus is $(0, 1)$ and the directrix is the x-axis, then $X(-1, 0)$. Let $x = my - 1$ be a line through X, so its slope is $1/m$. Then substitute in the equation, the quadratic in terms of y should have only one solution, which means it has discriminant zero. This is common trick for tangent to a conic.
- Problem 10: Consider the loci of points where $BC - AC$ is constant, a hyperbola; the maximum is when it degenerates to a line through the foci. Then it's just the distance between the foci.

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Session 10: Polynomials

October 13, 2017

Lecture problems

1. What is $(1+i)^{2017}$?
2. A robot's first move is to go east one unit. For its $n+1$st move, it turns $45°$ counterclockwise and travels half the distance of the nth move. How far is the robot from where it started after 2017 moves?
3. Complex numbers x, y, z satisfy $|x| = |y| = |z| = xyz = 1$ and $x+y+z = 0$. Find $|(2+x)(2+y)(2+z)|$.
4. (MMC) A third-degree polynomial satisfies $P(0) = -3$ and $P(1) = 4$. When $P(x)$ is divided by x^2+x+1 the remainder is $2x-1$. In the same division, what is the quotient?
5. (USAMO) Let a, b, c, d be real numbers such that $b-d \geq 5$ and the zeroes x_1, x_2, x_3, x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the minimum of $\left(x_1^2+1\right)\left(x_2^2+1\right)\left(x_3^2+1\right)\left(x_4^2+1\right)$.
6. Prove that $\sin(\pi/n)\sin(2\pi/n)\cdots\sin((n-1)\pi/n) = n/2^{n-1}$.
7. (QII3) Let $f(x)$ be a polynomial of degree 4 with integer coefficients, leading coefficient 1, and having $\sqrt{10}+\sqrt{11}$ as one of its zeroes. What is the sum of its coefficients?
8. (MMC) Find the x-coefficient of the fifth-degree polynomial satisfying $P(3^k) = k$ for $k = 0, 1, \ldots, 5$.
9. (AIME II 2005/13) Let $P(x)$ be a polynomial with integer coefficients satisfying $P(17) = 10$ and $P(24) = 17$. Given $P(n) = n+3$ has two distinct integer solutions, find their product.
10. (QI4) Suppose that r_1 and r_2 are the roots of the equation $4x^2 - 3x - 7 = 0$. What is the sum of the squares of the reciprocals of r_1 and r_2?
11. (Mandelbrot) Find the area of the triangle whose side lengths are the roots of $x^3 - 4x^2 + 5x - 1.9$.
12. (AIME 2005/8) Find the sum of the roots of the equation $2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$.

Completing the reals

- The algebraic completion of the reals are the complex numbers. That is, any polynomial with real coefficients has only complex roots. The complex numbers are algebraically closed.
- We can represent as $z = a+bi$ and think of it as Cartesian coordinates, or $re^{i\theta} = r(\cos\theta + i\sin\theta) = r\operatorname{cis}\theta$ and think of it as polar coordinates. The latter is often more helpful.
- We write $|z| = r$ and $\arg z = \theta$. Its conjugate is written as $\bar{z} = a - bi$ and has the important property $|z|^2 = z\bar{z}$. Conjugation distributes over arithmetic.
- Addition is vectorially or end-to-end. Multiplication is rotation and scaling: the radii multiply, the directions add because of $re^{i\theta}$. This gives de Moivre's theorem.
- Problem 1: It would be hard to expand, so instead use polar coordinates: $\left(\sqrt{2}\operatorname{cis}45°\right)^{2017}$.
- Problem 2: Represent each move with complex numbers, it's the previous move times $\frac{\sqrt{2}}{4}(1+i)$. It's a geometric series with first term 1 and 2017 terms. Compute its sum using problem 1.

Roots of unity

- The nth roots of unity are the n complex roots to $x^n - 1 = 0$. The square roots of unity are 1 and -1, the cube roots are 1, cis $\frac{2\pi}{3}$, and cis $\frac{4\pi}{3}$, the fourth roots are 1, i, -1, and $-i$. Generally, nth roots of unity are cis $\left(\frac{2\pi}{n}k\right)$ for nonnegative k less than n.
- Typically: $x^3 = 1$ but $x \neq 1$. Then if you let ω be the first cube root, ω^2 is the second cube root. Also $x^3 - 1 = (x-1)(x^2+x+1) = 0$, but $x \neq 1$, so $\omega^2 + \omega + 1 = 0$ as it is a root.
- Problem 3: If you see symmetric relations involving complex numbers, guess roots of unity. Which works here, luckily: $x = 1, y = \omega, z = \omega^2$ works. Then expand.

Polynomial division

- The division algorithm is $P(x) = Q(x)D(x) + R(x)$ with obvious relations between degrees. The remainder theorem and factor theorem follow from $D(x) = x - a$ and substitution.
- Problem 4: Write $P(x) = Q(x)(x^2+x+1) + (2x-1)$, then $Q(x)$ is linear, substitute $x = 0, 1$.

Factored form

- An nth degree polynomial can be written as $\prod(x - r_i)$, where r_i are its roots. This goes with substitution. Interpreting factors gives the table of signs, relative maxima and minima, etc.
- Problem 5: Thinking $P(x) = \prod(x - x_i)$ and factoring $\prod\left(x_i^2+1\right) = \prod(x_i - i)(x_i + i)$, we get the inspiration to write the product as $P(i)P(-i)$. The product is at least 16 from the inequality, and it's achievable when all the roots are 1.
- Problem 6: Let $\omega = e^{i\pi/n}$. Recall from session 5 that $\sin(k\pi/n) = (\omega^k - \omega^{-k})/2i$. Then the product is $\prod_{k=1}^{n-1} \frac{\omega^k - \omega^{-k}}{2i} = \frac{1}{2^{n-1}} \prod_{k=1}^{n-1} \frac{\omega^{-k}}{i}\left(1 - \omega^{2k}\right)$. The left factor is just $\omega^{-n(n-1)/2}$ divided by i^{n-1}, which cancels. The right factor is substituting 1 in the roots of unity polynomial $x^n - 1 = 0$, $x \neq 1$.

Root theorems

- Polynomials with real coefficients have conjugate non-real roots, polynomials with rational coefficients have conjugate irrational roots. Also, rational root theorem and synthetic division for root-finding.
- Problem 7: The roots are $\pm\sqrt{10} \pm \sqrt{11}$. Its sum of coefficients is $f(1)$, think of its factored form. Or, do $x = \sqrt{10} + \sqrt{11}$ and square twice.

Rewriting polynomials

- Problem 8: Consider the polynomial $P(3x) - P(x) - 1$. It is a fifth-degree polynomial with roots $k = 1, \ldots, 5$, so we can write it as $A(x-1)(x-2)\cdots(x-5)$ for some constant A. Substituting $x = 0$ gives the constant, and we only need to find the x-coefficient.
- Problem 9: We have $P(x) - x + 7 = A(x)(x-17)(x-24)$, also $P(x) - x - 3 = B(x)(x - n_1)(x - n_2)$. Then n_1 and n_2 are the roots of $A(x)(x-17)(x-24) = 10$, trial and error gives $n = 19, 22$.

Roots and coefficients

- Vieta's, rule of signs, discrete Fourier: "find the sum of the coefficients with exponent divisible by 3."
- Problem 10: Directly, $1/r_1^2 + 1/r_2^2 = (r_1 + r_2/r_1r_2)^2 - 2/(r_1r_2)$. There is also an important trick: for a bijection f, the substitution $x \to f^{-1}(x)$ changes the roots to $f(r_i)$. So here, we can try $x \to 1/x$.
- Problem 11: Recall Heron's and factored form. The semiperimeter is half the sum of the roots, or 2.
- Problem 12: Substitute $2^{111x} \to y$. The sum of x is the sum of $\frac{1}{111}\log_2 y$, so Vieta's on products.

VCSMS PRIME

Program for Inducing Mathematical Excellence

Session 11: Number Theory

October 20, 2017

Lecture problems

1. For how many integers $4 < d < 2017$ is 441 a perfect square in base d?
2. (AI16) Let N be a natural number whose base-2016 representation is ABC. Working now in base-10, what is the remainder when $N - (A + B + C + k)$ is divided by 2015, for some $k \in \{1, 2, \ldots, 2015\}$?
3. (AHSME 1993) Given $0 \le x_0 < 1$ define x_n for all positive integers n to be $2x_{n-1}$ if $2x_{n-1} < 1$, or $2x_{n-1} - 1$ otherwise. How many x_0 satisfy $x_0 = x_5$?
4. (AIME 1986) The sequence $1, 3, 4, 9, 10, \ldots$ consists of the positive integers which are powers of three or sums of distinct powers of three. Find its hundredth term.
5. (AIME 1985) Let $a_n = 100 + n^2$ for all positive integral n. Find the maximum GCD of a_n and a_{n+1}.
6. (HMMT 2002) Find the greatest common divisor of all numbers of the form $2002^n + 2$ for $n \in \mathbb{N}$.
7. (QI11) When $2a$ is divided by 7, the remainder is 5. When $3b$ is divided by 7, the remainder is also 5. What is the remainder when $a + b$ is divided by 7?
8. (QII2) Suppose that $159aa72$ is a multiple of 2016. What is the sum of its distinct prime divisors?
9. (Canada 2003) Find the last three digits of $2003^{2002^{2001}}$.
10. Factorize 89701, 160401 and $2^{18} + 1$.
11. (QI8) How many positive divisors of 30^9 are divisible by 400,000?
12. (AI12) Let $n = 2^{23}3^{17}$. How many factors of n^2 are less than n but do not divide n?
13. (QIII5) Let $s(n)$ be the number of terminal zeroes in the decimal representation of $n!$. How many positive integers less than 2017 cannot be expressed in the form $n + s(n)$ for some positive integer n?
14. How many ordered pairs of positive unit fractions have sum $\frac{1}{6}$?

Digits and bases

- Problem 1: All of them: $441_d = 4d^2 + 4d + 1 = (2d + 1)^2$ in base 10.
- Problem 2: $N = C + 2016B + 2016^2 A$ in base 10. Then $N - (A + B + C + k) = 2015B + (2016^2 - 1)A - k$, but the first two terms are divisible by 2015, so the remainder is $2015 - k$.
- Problem 3: The doubling makes us consider binary. The sequence moves the decimal point to the right.
- Problem 4: Writing in ternary, these are the numbers that only have 0 or 1, so the terms are just binary.

Divisibility

- Note $a|b$ iff $(a, b) = a$. We have $(a, b) = (a - b, b)$. This gives the Euclidean algorithm.
- Problem 5: We have $(100+n^2, 100+(n+1)^2) = (100+n^2, 2n+1) = (200+2n^2, 2n+1) = (200-n, 2n+1) = (400 - 2n, 2n + 1) = (401, 2n + 1)$.
- Problem 6: Equivalent to GCD of 2004 and $2002^n - 2002 = 2002(2002^{n-1} - 1)$. But it is well-known that $(a^n - 1, a^m - 1) = a^{(n,m)} - 1$.

Modulo

- The integers modulo a prime form a field. Thus we have arithmetic. For composite moduli, CRT. From most to least common, know how to solve linear systems, Fermat's Little, Euler Totient, Wilson, Lucas.
- Problem 7: $2a \equiv 5 \pmod 7$ implies $a \equiv 5 \cdot 2^{-1} \pmod 7$, but the inverse of 2 is 4 (trial-and-error, or $2x + 7y = 1$.), so $a \equiv 5 \cdot 4 \equiv 6 \pmod 7$. Similarly $b \equiv 4 \pmod 7$ and $a + b \equiv 3 \pmod 7$.
- Problem 8: $159aa72 \equiv 1590072 + 1100a \equiv 1464 + 1100a \equiv 0 \pmod{2016}$. Not hard to find a.
- Problem 9: Split into mod 8 and mod 125. Binary exponentiation for 3^{52} mod 125.

Factorization

- Usually involves clever algebraic manipulation. If you end up with a number not in an obviously factorable way, always try to rewrite as difference of two squares.
- Problem 10: The first is $300^2 - 300 + 1 = (300 + 1)^2 - 900$. The second is $20^4 + 20^2 + 1 = (20^2 + 1)^2 - 20^2$. By Sophie–Germain, the last is $(1 - 2^5 + 2^9)(1 + 2^5 + 2^9)$.

Multiplicative number theory

- A function f is multiplicative if $f(mn) = f(m)f(n)$ for all $(m, n) = 1$. Such a function is determined completely by powers of primes. Examples: $\tau(n)$, number of divisors; $\sigma(n)$, sum of divisors; $\phi(n)$, number of positive integers less than and relatively prime to n.
- Problem 11: Answer is just $\tau(30^9/400{,}000)$.
- Problem 12: Each factor of n^2 has a pair that multiplies to n^2; one is smaller and one is larger than n, so $\frac{1}{2}\left(\tau(n^2) - 1\right)$ is the number of factors less than n. These overcount the factors of n, subtract $\tau(n)$.
- "Sum of divisors that are perfect squares", "that are even", "difference of divisors with odd sum of exponents and even sum of exponents", "product of divisors" or "sums of product of non-zero digits".

Valuation

- $\nu_p(n)$ is the largest power of p that divides n. Most important is $\nu_p(n!)$, which is given by de Polignac, $\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \cdots$. This allows us, for example, to compute the valuation for binomial coefficients. This is also equivalent to $(p - 1)\nu_p(n!) = n - s_p(n)$ where $s_p(n)$ is the sum of the digits of n in base p.
- Problem 13: The limiting factor of 10 is 5, so $s(n) = \nu_5 n!$. The "bumps" happen at multiples of 5. Alternatively, consider base 5 and use the second version of de Polignac's.

Diophantine equations

- The building block is $ax + by = c$ for fixed a, b, c and integers x, y. Bezout's: only need to solve the case $ax + by = (a, b)$. Then we only need to find one solution by using Euclidean algorithm or trial-and-error.
- Frobenius: if $(a, b) = 1$, for nonnegative x, y, the number of positive integers that can't be written as $ax + by$ is $\frac{1}{2}(a - 1)(b - 1)$. Chicken McNugget: the largest that can't is $ab - a - b$.
- Finally, factorization, often SFFT, will take care of most other cases.
- Problem 14: Equation $\frac{1}{a} + \frac{1}{b} = \frac{1}{6}$ is equiv to $6a + 6b = ab$, or $(6 - a)(6 - b) = 36$. Casework on factors.