VCSMS PRIME

Program for Inducing Mathematical Excellence
Session 3: Combinatorial Principles
September 21, 2017

Lecture problems

1. (QII1) I have 2016 identical marbles. I plan to distribute them equally into one or more identical containers. How many ways can this be done if I have an unlimited number of containers?
2. (QI13) How many three-digit numbers have distinct digits that add up to 21?
3. (QIII2) Using the numbers $1, 2, 3, 4, 5, 6$ and 7, we can form $7! = 5040$ 7-digit numbers in which the 7 digits are all distinct. If these numbers are listed in increasing order, find the 2016th number in the list.
4. A set of points is chosen on the circumference of a circle so that the number of different triangles with all three vertices among the points is equal to the number of pentagons with all five vertices in the set. How many points are there?
5. Compute $\binom{1000}{0} + \binom{1000}{2} + \binom{1000}{4} + \cdots + \binom{1000}{1000}$.
6. (AI15) How many numbers between 1 and 2016 are divisible by exactly one of $4, 6$, or 10?
7. (QII9) How many ordered triples of positive integers (x, y, z) are there such that $x + y + z = 20$ and two of x, y, z are odd?

Counting is multiplication

- The number of ways to choose in a series of independent choices is the product of the number of choices. When it's in a row (or a circle, or whatever) it's called a *permutation*.
- Problem 1: The formula for the number of divisors is $\prod(e_i + 1)$ because for each prime p_i, a divisor can have either $p_i^0, p_i^1, p_i^2, \ldots, p_i^{e_i}$ as a factor.
- Problem 2: Sometimes you just have to count brute force. Then multiply by 3! to account for the number of arrangements, but be careful with zero. General rule: $n!$ ways to arrange n items in a row.
- Problem 3: This is like "the permutations of VALMASCI are sorted alphabetically, what is the nth word in the list?" This is called *lexicographical order*, and you count by going left-to-right, slot-by-slot. How many with first digit 1? First digit 2? Then if you go over 2016 you move to the next digit.

Counting is division

- When order matters, it's a permutation, but when it doesn't it's a combination. Sometimes we *overcount*, and we want to count things as the same. How many ways to arrange letters in PHILIPPINES?
- Permutations on a circle: divide by n. With reflections: divide by 2.
- When order doesn't matter, it's called *combination*: divide by $k!$. This gives binomial coefficients $\binom{n}{k}$.
- Problem 4: A triangle is defined by its three vertices, same for a pentagon, so $\binom{n}{3} = \binom{n}{5}$. Cancel.
- Why "binomial coefficient"? Because it's the coefficients in $(x + y)^n$.
- Problem 5: What happens when we expand $(1 + 1)^{1000}$ and $(1 - 1)^{1000}$ using the binomial theorem?

Counting is addition (and subtraction)

- Draw Venn diagrams and add and subtract. For PIE, the pattern is add, subtract, add, subtract, add, subtract.
- Problem 6: Add divisble by $4, 6, 10$, subtract twice divisible by $\text{lcm}(4, 6) = 12, \text{lcm}(4, 10) = 20, \text{lcm}(6, 10) = 30$, add thrice divisible by $\text{lcm}(4, 6, 10) = 60$.

Counting is bijection

b balls	u urns	no restriction	at most 1	at least 1
D	D			(hard)
I	D			
D	I	(hard)		(hard)
I	I	(hard)		(hard)

- Twelvefold way: distinguishable vs indistinguishable balls (2), distinguishable vs indistinguishable urns (2), no restriction vs at most one ball in each urn vs at least one ball in each urn (3).
- We know how to do five out of twelve:
  - How many ways to give three different candies to five children? (5^3)
  - How many ways to give three different candies to five children, so that each child gets at most one candy? ($5 \times 4 \times 3$)
  - How many ways to give three identical candies to five children, so that each child gets at most one candy? ($\binom{5}{3}$)
  - How many ways to place three different candies in five identical pouches, so that each pouch has at most one candy? (1)
  - How many ways to place three identical candies, in five identical pouches, so that each pouch has at most one candy? (1)
- Another five out of twelve have no simple formula, and should be done manually.
- The remaining two can be done with balls and urns arguments:
  - How many ways to give three identical candies to five children? ($\binom{3+5}{3}$)
  - How many ways to give three identical candies to two children, so each child gets at least one candy? ($\binom{3-1}{2-1}$)
- Appears in lots of ways:
  - $w + x + y + z = c$ for nonnegative or positive integers.
  - How many ways to roll 5 dice so the sum is 16?
  - Problem 7: There are $\binom{3}{2}$ ways to pick the two odd numbers. Rewrite as $2a + 1, 2b + 1, 2c$, and then do balls and urns, then remember to multiply by $\binom{3}{2}$.
  - Selecting 6 cookies out of 3 different kinds, when you have unlimited supplies of each kind.
  - Number of 7-digit numbers with digits in increasing order.
  - How many ways to arrange 3 maple trees, 4 oak trees and 5 birch trees in a row so no two birch trees are adjacent?