VCSMS PRIME

Program for Inducing Mathematical Excellence
Session 5: Trigonometry
September 28, 2017

Lecture problems

1. (AIME 1995/7) Given $(1+\sin t)(1+\cos t) = \frac{5}{4}$, find $(1-\sin t)(1-\cos t)$.
2. (QI9) Evaluate the following sum: $1 + \cos\frac{\pi}{3} + \cos\frac{2\pi}{3} + \cos\frac{3\pi}{3} + \cdots + \cos\frac{2016\pi}{3}$.
3. (Huang[1]) Divide $\sin 3x\,(2\cos 2x - 1)$ by $\sin x\,(2\cos 4x + 1)$ and simplify.
4. (AIME 1996/10) Find the smallest positive integer solution to $\tan 19x^\circ = \dfrac{\cos 96^\circ + \sin 96^\circ}{\cos 96^\circ - \sin 96^\circ}$.
5. (AI6) Find the exact value of $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right)$.
6. (Morrie's Law) Simplify $\cos 20^\circ \cos 40^\circ \cos 80^\circ$.
7. Find the exact value of $\cos \pi/5$.
8. (PEM 2016/10) Find the ratio of $\sin 1^\circ + \sin 2^\circ + \cdots + \sin 44^\circ$ to $\cos 1^\circ + \cos 2^\circ + \cdots + \cos 44^\circ$.
9. (Stewart's Theorem) In triangle ABC, point D is on line BC. Let $AD = d, BD = m$ and $CD = n$. Then $man + dad = bmb + cnc$.

As ratios of sides

- In terms of triangles: sine, cosine, tangent, secant, cosecant, cotangent, are ratios of sides. Example: find $\tan\alpha$ given $\sin\alpha = \frac{1}{2}$. This is the classical development in classroom math.
- Values for special angles: $30^\circ, 45^\circ, 60^\circ$ can be derived from special right triangles $30^\circ - 60^\circ - 90^\circ$ (equilateral triangle), and $45^\circ - 45^\circ - 90^\circ$ (square). We can get 0° and 90° from reasoning.
- Identities: cofunctions, Pythagorean identities.
- Problem 1: Abuse symmetry: replace sums and products with variables. Pythagorean identity.

As lengths in a circle

- In terms of circles: the functions are lengths in the unit circle: $(\sin\theta, \cos\theta)$. Tangent is the length of the tangent to x-axis, cotangent is length to y-axis. Secant and cosecant are x and y intercepts.
- Problem 2: Radians are more natural than degrees: 2π radians is 360°.
- Identities: reflection over $0, \frac{\pi}{4}, \frac{\pi}{2}$, shifts by $\frac{\pi}{2}, \pi, 2\pi$, Pythagorean identities, $\csc^2\theta + \sec^2\theta = (\cot\theta + \tan\theta)^2$.

As complex numbers

- Euler's identity says $e^{ix} = \cos x + i\sin x$. Cosine is the real part and sine is the imaginary part.
- If $z = e^{ix}$ then $1/z = e^{-ix}$. We can state $\cos x$ and $\sin x$ in terms of z only. Also, by de Moivre, $\left(e^{ix}\right)^n = \cos nx + i\sin nx$. We can find $\cos nx$ and $\sin nx$ in terms of z only.
- Identities: reflection, Pythagorean identities.
- Problem 3: Let $z = e^{ix}$. Substitute sine and cosine. Everything cancels nicely.

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[1]From Complex Numbers in Trigonometry, `https://aops.com/community/c6h609795`. Read it, it's good.

Sum and difference

- Fundamental is $\sin(x+y)$. Many derivations: stick two right triangles with angles x, y and common side and find the area; use the unit circle and rotate; Euler's identity.
- Derive $\sin(x \pm y), \cos(x \pm y), \tan(x \pm y), \cot(x \pm y)$. Triple tangent formula: $x + y + z = \pi$ iff $\tan x + \tan y + \tan z = \tan x \tan y \tan z$.
- Problem 4: Inspired by tangent sum formula. Force tangent by dividing by $\cos 96°$ and $\tan 45° = 1$.
- Inverses for trigonometric functions exist. Derive $\tan^{-1} x \pm \tan^{-1} y$, $\cot^{-1} x \pm \cot^{-1} y$ by subtituting in the sum and difference formulas.
- Problem 5: The shortcut for $\tan^{-1}(p_1/q_1) + \tan^{-1}(p_2/q_2)$ is "cross multiply and add for numerator, product of denominators minus product of numerators for denominator."
- Derive $\sin 2x$. Write $\cos 2x$ in three ways using Pythagorean identity, and use that to derive half-angle formulas. (Sine is minus, same as in complex numbers.) There is also derivation with Euler's formula.
- Problem 6: The doubling inspires us to force double angle formula. Let expression be x and multiply both sides by $2\sin 20°$. (Interestingly, $\tan 20° \tan 40° \tan 80° = \tan 60°$.)
- Problem 7: This is important. Let $a = \cos \pi/5$ and $b = \cos 2\pi/5$. Then use double angle formulas on a and b, but $\cos 4\pi/5 = -\cos \pi/5 = -a$.

Prosthaphaeresis

- Greek *prosthesis* means addition, *aphaeresis* means subtraction. Which is how you derive them: $\cos x \cos y, \sin x \sin y, \sin x \cos y$ by cancelling out $\cos(x+y)$ and $\cos(x-y)$, etc.
- Reverse formulas: find $\sin x \pm \sin y$ by reversing the prosthaphaeresis formulas. As in, let $x' = x - y$ and let $y' = x + y$ and rewrite.
- Problem 8: Looks like arithmetic sequence. We did arithmetic sequence by pairing up opposite terms. Here, we pair $\sin 1°$ and $\sin 44°$ and use sum-to-product, etc. Everything cancels.

Laws

- Extended law of sines: draw the circumradius and use the definition of sine. Law of cosines: drop an altitude use the Pythagorean theorem twice.
- Problem 9: Apply law of cosines twice: on $\triangle ABD$ and $\triangle BCD$.

As functions

- Sine and cosine: domain is $\mathbb{R}$, range is $\{-1, 1\}$. Period 2π. Graphs are translations of each other.
- Cosecant and secant graphs are like a bunch of parabolas. Cosecant domain is $\mathbb{R} - \{n\pi, n \in \mathbb{Z}\}$ and secant domain is $\mathbb{R} - \{(2n+1)\pi/2, n \in \mathbb{Z}\}$. Range is $(-\infty, -1] \cup [1, \infty)$.
- Tangent and cotangent: tangent domain is $\mathbb{R} - \{(2n+1)\pi/2, n \in \mathbb{Z}\}$ and cotangent domain is $\mathbb{R} - \{n\pi, n \in \mathbb{Z}\}$. Range is $\mathbb{R}$.
- Inverse functions: arcsin and arccos are domain $[-1, 1]$. Arcsin has range $\{-\pi/2, \pi/2\}$, arccos has range $\{0, \pi\}$. Arctan has domain $\mathbb{R}$ and range $\{-\pi/2, \pi/2\}$: it maps the whole real line onto a finite interval.
- This means we need to be careful in solving equations like $\sin x = 1/2$, because there are infinitely many solutions. Also, $\sin^{-1} x + \cos^{-1} x = \pi/2$ and stuff like $\sin(\cos^{-1} x) = \sqrt{1 - x^2}$.

deg	rad	sin	cos	tan
0°	0			
15°	$\frac{\pi}{12}$			
18°	$\frac{\pi}{10}$			
22.5°	$\frac{\pi}{8}$			
30°	$\frac{\pi}{6}$			

deg	rad	sin	cos	tan
36°	$\frac{\pi}{5}$			
45°	$\frac{\pi}{4}$			
60°	$\frac{\pi}{3}$			
90°	$\frac{\pi}{2}$			
180°	π			

Pythagorean identity: $\sin^2\theta + \cos^2\theta =$ ______

Dividing by $\sin^2\theta$ gives: ____________________

Dividing by $\cos^2\theta$ gives: ____________________

Sum and difference:

$\sin(x \pm y) =$ ______________________________

$\cos(x \pm y) =$ ______________________________

$\tan(x \pm y) =$ ______________________________

$\cot(x \pm y) =$ ______________________________

Arctans:

$\tan^{-1} x + \tan^{-1} y =$ _________________________

$\cot^{-1} x + \cot^{-1} y =$ _________________________

$\tan^{-1}\frac{p_1}{q_1} + \tan^{-1}\frac{p_2}{q_2} =$ _______________________

Half-angle:

$\sin\frac{x}{2} =$ ______________________________

$\cos\frac{x}{2} =$ ______________________________

Double angle:

$\sin 2x =$ ______________________________

$\cos 2x =$ ______________________________

$\tan 2x =$ ______________________________

$\cot 2x =$ ______________________________

Product-to-sum:

$2\cos x \cos y =$ ___________________________

$2\sin x \sin y =$ ___________________________

$2\sin x \cos y =$ ___________________________

Sum-to-product:

$\sin x \pm \sin y =$ ___________________________

$\cos x + \cos y =$ ___________________________

$\cos x - \cos y =$ ___________________________

Law of sines: ___________________________

Law of cosines: _________________________.