VCSMS PRIME

Program for Inducing Mathematical Excellence

Session 6: Circles and Polygons

September 29, 2017

Lecture problems

1. (AI3) Let O be the circumcenter of acute triangle ACD. The tangent to the circumcircle at A intersects line CO at B. If $AB||CD$, $AB = 6$ and $BC = 12$, what is the length of CD?
2. (QI5) The sides of a triangle are 3, 5 and x. The sides of another triangle are 4, 6 and y. If the sides of both triangles are integers, what is the maximum value of $|x - y|$?
3. Two poles of heights x and y are perpendicular to the ground. A rope is tied from the top of the first to the bottom of the second, and another rope is tied from the top of the second to the bottom of the first. The two ropes intersect at a point that is distance z from the ground. Show $1/x + 1/y = 1/z$.
4. A square is inscribed in triangle ABC such that two vertices lie on side BC, one vertex lies on AB, and the remaining vertex lies on AC. If $AB = 6$, $BC = 7$, and $CA = 8$, find the area of the square.
5. A point P and a rectangle $ABCD$ satisfy $AP = 7$, $BP = 5$, and $CP = 8$. Find DP.
6. (QI3) One diagonal of a rhombus is three times as long as the other. If the rhombus has an area of 54 square meters, what is its perimeter?
7. (AI5) In parallelogram $ABCD$, $AB = 1$, $BC = 4$, and $\angle ABC = 60°$. Supose that AC is extended from A to a point E beyond C so that ADE has the same area as the parallelogram. Find the length of DE.
8. The diagonals in trapezoid $ABCD$ with $AB||CD$ intersect at point E. Given that $[ABE] = 25$ and $[CDE] = 64$, find $[ABCD]$.
9. A quadrilateral circumscribed about a circle has side lengths $1, 2, 3, x$ in order. Find x.
10. Two regular pentagons $WORLD$ and $LAYER$ are constructed outside each other. Find the length of RL given $[ODAE] = \sqrt{5}$.

Circles

- The perpendicular bisector of a chord is a diameter. Power of a point: If chords AB and CD intersect internally at P (if P lies outside the circle it must be closer to A than B, and to C than D), then $AP \cdot BP = CP \cdot DP$.
- Problem 1: Let AO intersect CD at E. Then $\triangle COE \sim \triangle BOA$, use power of a point.
- We can find lengths of common external and internal tangents given distance of centers and radii.

Triangles

- Problem 2: By triangle inequality, $2 < x < 8$ and $2 < y < 10$. Largest difference when $x = 3, y = 9$.
- All you have to do is find similar triangles! Example: altitude of a right triangle in terms of a, b, c.
- Problem 3: Look at the bases. Add up ratios: $z/x + z/y = 1$. This comes up a lot.

Squares

- Problem 4: Divide the area of the triangle into the area above the square, the area of the square, and the area to the left and right of the square. You get side length of square as $bh/(b + h)$.

Rectangles

- A parallelogram with congruent diagonals, or one right angle.
- Problem 5: British Flag Theorem states that $AP^2 + CP^2 = BP^2 + DP^2$.

Rhombi

- A quadrilateral with all sides the same length. Its diagonals meet at right angles.
- Problem 6: Let diagonals be $6x$ and $2x$, its area is $6x \cdot 2x/2$. Then by Pythagorean, a side is $\sqrt{(3x)^2 + x^2}$.

Parallelograms

- A quadrilateral whose opposite sides are parallel. Also, a quadrilateral whose diagonals bisect each other. Also, a quadrilateral whose pairs of opposite sides are the same length. Lots of congruent triangles.
- Parallelogram law: sum of the squares of the sides is equal to sum of the squares of the diagonals.
- Problem 7: The area of ADE is twice the area of ACD, but they both have the same height, so AE is twice AC. Use law of cosines and Stewart's theorem.

Trapezoids

- A quadrilateral with *at least* (sometimes exactly) one pair of opposite sides.
- If $AB||CD$ and AD, BC intersect at E, then $\triangle ABE \sim \triangle DCE$ and $[AED] = [BEC]$. The segment connecting the midpoints of AC and BD passes through the midpoints of the diagonals and has length $(AB + CD)/2$. The segment connecting midpoints of diagonals has length $|AB - CD|/2$.
- A trapezoid is isosceles iff it is cyclic. If you have a cyclic quadrilateral with congruent opposite sides, then it's automatically a trapezoid.
- Problem 8: From $[ABE][CDE] = [AED][BEC]$ (which is true for any quadrilateral) and $[AED] = [BEC]$ we can get the answer. Generally $\sqrt{[ABCD]} = \sqrt{[ABE]} + \sqrt{[CDE]}$ for any trapezoid.

Quadrilaterals

- The quadrilateral joining the midpoints of the sides is a parallelogram, and has half the area.
- The diagonals AC and BD are perpendicular iff $AB^2 + CD^2 = BC^2 + DA^2$. Then $[ABCD] = AC \cdot BD/2$.
- If cyclic, area is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$. This is Brahmagupta's formula.

Tangential polygons

- Use the fact that the tangents to the same circle from the same point are the same length. Example: tangent lengths to the incircle. Another: can a pentagon with side lengths $2, 3, 4, 5, 6$ be tangential?
- Problem 9: By Pitot's theorem, $1 + 3 = x + 2$, so $x = 2$.

Polygons

- Polygon inequality: any side of a polygon has to be less than the sum of the other sides. If this is satisfied, you can imagine it as a wireframe and can shift it to force a lot of things.
- Don't forget MMC: how many diagonals in an n-gon? Angle of regular n-gon?
- Problem 10: I told you the trigonometric functions of $36°$ were important. Here: $\cos 36° = \frac{1}{4}\left(\sqrt{5} + 1\right)$ and $\cos 72° = \frac{1}{4}\left(\sqrt{5} - 1\right)$. Use $\cos a = -\cos(180° - a)$. Now law of cosines and bash.
- Whoops need to talk about three-dimensional geometry and $v - e + f = 2$.