Program for Inducing Mathematical Excellence

Final Exam

27 October 2017

PART I. Choose the best answer. Each correct answer is worth two points.

1. Which of the following choices is the largest?

   (a) 121^6 (b) 12^{12} (c) 5^{18} (d) 2^{42}

2. In triangle ABC, $AB = 6$ and $BC = CA = 5$. Which of the following statements are false?
   (a) Triangle ABC is acute, but if $AB = 8$ instead, it would be obtuse.
   (b) Of the triangle's angles, the one with the largest measure is $\angle BCA$.
   (c) The value of $\tan \angle ABC$ is greater than 1.5 but less than 2.
   (d) The area of triangle ABC is numerically equal to three-fourth its perimeter.

3. Ankan flips six fair coins. What is the probability he gets more than three heads?

   (a) $\frac{11}{32}$ (b) $\frac{1}{2}$ (c) $\frac{21}{32}$ (d) $\frac{13}{16}$

4. Each of 6 people picks another person among them as their giftee, and no two people pick the same giftee. Everyone holds a nametag with their name on it. In a single move, each person gives the nametag they're holding to their giftee. After several moves, everyone holds their own nametag again. What is the minimum number of moves needed to ensure this?

   (a) 12 (b) 60 (c) 120 (d) 144

5. Points I, A, and N are outside regular octagon $PROBLEMS$ such that $PIANO$ is a regular pentagon. What is $m\angle ROA$?

   (a) 49.5° (b) 51° (c) 52.5° (d) 54°

6. John is answering a question on his exam: "What interval is 50% to 65% of the maximum heart rate of a 20-year-old?" John is given the following choices, and he knows the correct answer is one of these. Which of these is the correct answer?

   (a) $[100, 130]$ bpm (b) $[110, 140]$ bpm (c) $[120, 150]$ bpm (d) $[130, 160]$ bpm

7. Given that $\sin x + \cos x = \frac{4}{3}$, what is the value of $\sin^3 x + \cos^3 x$?

   (a) $\frac{29}{54}$ (b) $\frac{17}{27}$ (c) $\frac{13}{18}$ (d) $\frac{22}{27}$

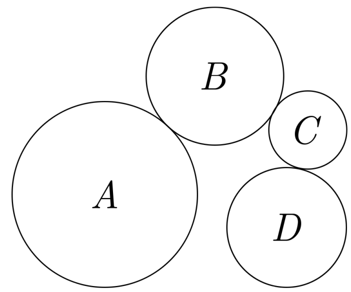
8. Class A has a boys and b girls, while Class B has x boys and y girls, where a, b, x and y are integers such that $0 < a < x < b < y$. Each pupil in Class A is paired with someone from Class B. No pupil is in two different pairs. What is the maximum number of pairs with one boy and one girl?

(a) $a + x$ (b) $x + b$ (c) $b + y$ (d) $y + a$

9. Suppose the roots of $x^3 + 20x + 17 = 0$ are a, b and c. What is $a^3 + b^3 + c^3$?

(a) -34 (b) -37 (c) -40 (d) -51

10. In the figure on the right (not drawn to scale), four gears are shown, represented as circles. Gear A has radius 6, gear B has diameter 6, gear C has circumference 6, and gear D has area 6. The first gear is turned 90°. By how many radians does gear D turn?



(a) $\frac{\pi^2}{2}$ (b) $\frac{\pi\sqrt{6\pi}}{2}$ (c) $\frac{\pi^2}{4}$ (d) $\frac{\pi\sqrt{6\pi}}{4}$

11. Given that a, b and c are positive real numbers satisfying

$$\begin{aligned} ab + 2a + b &= 13, \\ bc + 3b + 2c &= 36, \\ ca + c + 3a &= 32, \end{aligned}$$

then a can be written as $\frac{p + q\sqrt{r}}{s}$ for some integers p, q, r and s, where r is not divisible by the square of a prime and p, q and s have greatest common divisor 1. What is $p + q + r + s$?

(a) 5 (b) 7 (c) 9 (d) 11

12. Each vertex of a regular pentagon is colored either pink, red, indigo, mauve or ebony. Two colorings are the same if one can be rotated to obtain the other. Find the number of different colorings.

(a) $5^4 - 1$ (b) 5^4 (c) $5^4 + 4$ (d) $5^4 + 5$

13. Which of the following choices is nearest to $\sqrt{3} + \sqrt{8} + \sqrt{15} + \sqrt{24} + \sqrt{35} + \sqrt{48}$?

(a) 26.2 (b) 26.4 (c) 26.6 (d) 26.8

14. Five husbands and their five wives sit in a row. Exactly four wives are sitting next to their respective husbands. In how many ways is this possible?

(a) 40 320 (b) 44 160 (c) 80 640 (d) 84 480

15. What is the remainder when $20^{17} + 17^{20}$ is divided by 21?

(a) 3 (b) 5 (c) 15 (d) 17

PART II. Choose the best answer. Each correct answer is worth three points.

1. There exists distinct nonzero digits a, b and c such that the eight-digit number $\overline{aaabbbb5}$ is the square of the four-digit number $\overline{ccc5}$. There are two possible values for a. What is their sum?

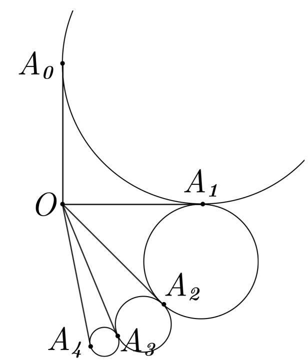
   (a) 5 (b) 9 (c) 13 (d) 17

2. In the figure on the right, $OA_0 = OA_1 = 1$, and $\angle A_0OA_1 = 90°$. First, Bobby draws a circle tangent to OA_0 at A_0 and to OA_1 at A_1. For each positive integer n, Bobby then draws:

   a) a ray OA_{n+1} such that $2\angle A_nOA_{n+1} = \angle A_{n-1}OA_n$, and

   b) a circle tangent to OA_n at A_n and to OA_{n+1} at A_{n+1}.

   Which of the following choices is closest to the sum of the areas of the infinitely many circles Bobby draws?

   (a) 1.17π (b) 1.23π (c) 1.29π (d) 1.35π

3. Two circles, both radius $\sqrt{2}$, have centers $\sqrt{3}-1$ apart. What is the area of their common region?

   (a) $\frac{5\pi}{3} - 1$ (b) $2\pi - \frac{1}{2}$ (c) $\frac{4\pi}{3} - \frac{1}{2}$ (d) $\frac{7\pi}{3} - \frac{\sqrt{3}}{2}$

4. Four people are trying to guess the location of a point X in the Cartesian plane. The guess closest to X was $(1,0)$, the second-closest was $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$, the third-closest was $(0,0)$, and the farthest was $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. Given this information, what is the area of the region where point X can be located?

   (a) $\frac{\sqrt{3}}{8}$ (b) $\frac{\sqrt{3}}{6}$ (c) $\frac{\sqrt{3}}{4}$ (d) $\frac{\sqrt{3}}{3}$

5. For a positive integer n, let $f(n) = -1$ if the sum of the exponents in its prime factorization is odd, and $f(n) = 1$ if it is even. For example, $f(1) = 1$, $f(5) = -1$, and $f(24) = f\left(2^3 \cdot 3\right) = 1$ since $3+1$ is even. What is the sum of $df(d)$ over all positive factors d of 6480?

   (a) $-22\,506$ (b) -2684 (c) 2684 (d) $22\,506$

6. For how many integers $0 \le k \le 2017^2$ does 2017 divide $\binom{2017^2}{k}$?

   (a) $2017^2 - 2017$ (b) $2017^2 - 2016$ (c) $2017^2 - 1$ (d) 2017^2

7. Polynomial $P(x)$ has rational coefficients, degree 4032, and leading coefficient 2017^2. The numbers $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \ldots, \frac{1}{\sqrt{2017}}$ are 2016 of its roots. What is the sum of its coefficients?

   (a) $\frac{1}{2017}$ (b) 0 (c) 1 (d) 2017

8. Evaluate the sum

$$\frac{\log_{2^2} 1 - 1}{\log_{2^2} 2017 + \log_{2^2} 2017!} + \frac{\log_{3^2} 2 - 1}{\log_{3^2} 2017 + \log_{3^2} 2017!} + \cdots + \frac{\log_{2017^2} 2016 - 1}{\log_{2017^2} 2017 + \log_{2017^2} 2017!},$$

where the numerator is $\log_{n^2}(n-1) - 1$ and the denominator is $\log_{n^2} 2017 + \log_{n^2} 2017!$, the sum ranging from $n = 2$ to 2017.

(a) -2 (b) -1 (c) 0 (d) 1

9. Trapezoid $ABCD$ has AB parallel to CD, $AB = 25$, $AC = 20$ and $AD = BC = 15$. Find its area.

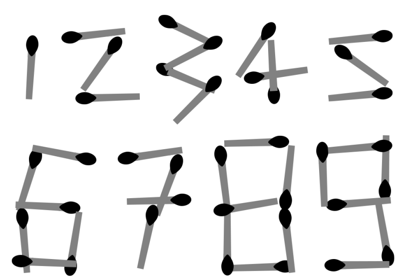
(a) 128 (b) 144 (c) 180 (d) 192

10. How many ways are there to tile a 3×15 rectangle with 3×1 and 1×3 tiles? Rotations and reflections are counted as different tilings.

(a) 162 (b) 171 (c) 180 (d) 189

PART III. All answers should be in simplest form. Each correct answer is worth six points.

1. Chris has 2018 matchsticks, and forms several digits with them, using the pattern shown in the figure on the right. What is the largest possible sum of the digits Chris can make?



2. In triangle ABC, let A_B and A_C be the reflections of A with respect to B and C respectively. Define B_A, B_C, C_A and C_B similarly. Find the ratio of the area of the hexagon $A_BA_CB_CB_AC_AC_B$ to the area of triangle ABC.

3. What is the sum of the prime factors of 3 200 021?

4. The function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(1) = 0$ and

$$\frac{f(x+y) + f(x-y)}{x} - 4x^2 = \frac{f(x+y) - f(x-y)}{y} - 4y^2$$

for all nonzero x and y. Find the value of $f(20) - f(17)$.

5. It is possible to write every positive integer uniquely as a *proper sum* like so:

$$8 = 1 + 1 + 2 + 4 \qquad 81 = 1 + 2 + 2 + 4 + 8 + 16 + 16 + 32 \qquad 15 = 1 + 2 + 4 + 8$$

A proper sum has consecutive powers of two, starting from 1. Every power of two in a proper sum appears either once or twice. The following sums are *not* proper:

$$13 = 1 + 2 + 2 + 8 \qquad 10 = 2 + 4 + 4 \qquad 7 = 1 + 2 + 2 + 2$$

Let $f(n)$ be the number of powers of two that appear twice in the proper sum of n. For example, $f(81) = 2$, since 2 and 16 appear twice. Find the 70th smallest positive integer n with $f(n) = 3$.