This is the practice test, due on Friday, October 27, 2017.

- The test follows the qualifying stage format, which means the time limit is two hours. These should be continuous; do not work for one hour on one day and then one hour on another day.
- No aids except scratch paper, ruler, and compass are permitted. No graph paper, protractors, calculators, computers, or mobile phones are allowed. Do not leave the testing area for the duration of the exam.
- Write your answers legibly on a clean sheet of short bond paper, with your name.
- Do not discuss the problems before Friday, October 27, when they are due.

Do not turn the page until you have a timer for two hours and are ready to answer **without interruptions**.

Program for Inducing Mathematical Excellence

Practice Final Exam

27 October 2017

PART I. Choose the best answer. Each correct answer is worth two points.

1. Let m be a positive integer less than 2015. What is the largest possible remainder when 2015 is divided by m?

   (a) 1004 (b) 1005 (c) 1006 (d) 1007

2. Compute $\sqrt{(31)(30)(29)(28)+1}$.

   (a) 701 (b) 755 (c) 811 (d) 869

3. Three fair six-sided dice are labeled $\{1,2,3,4,5,6\}$, $\{1,2,3,4,5,6\}$, and $\{1,2,3,7,8,9\}$. All three dice are rolled. What is the probability that at least two have the same value?

   (a) $\frac{1}{4}$ (b) $\frac{5}{18}$ (c) $\frac{11}{36}$ (d) $\frac{1}{3}$

4. Suppose w, x, y, z are real numbers greater than 1 such that $\log_x w = 24, \log_y w = 40$ and $\log_{xyz} w = 12$. Find $\log_z w$.

   (a) 15 (b) 30 (c) 60 (d) 120

5. Let $ABCD$ be a square with side length 100, and let M be the midpoint of AB. Point P is selected inside $ABCD$ such that $MP = 50$ and $PC = 100$. Find AP^2.

   (a) 1400 (b) 1600 (c) 1800 (d) 2000

6. Let the four real solutions to the system $x^2 + 6y - 36 = 0$ and $y^2 - 6x - 36 = 0$ be (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4). What is $(y_1 + y_2 + y_3 + y_4) - (x_1 + x_2 + x_3 + x_4)$?

   (a) 0 (b) 6 (c) 18 (d) 36

7. There exists three-digit numbers A, B and C such that moving the last digit of A to its beginning gives B, and moving the last digit of B to its beginning gives C. Both A and B are perfect squares, but C is not. What is the sum of C's digits, given it is not a perfect square?

   (a) 7 (b) 10 (c) 13 (d) 16

8. What is the sum of the coefficients in the expansion of $(x+y)^{10} + (x-y)^{10}$?

   (a) 512 (b) 513 (c) 1024 (d) 1025

9. What are the last two digits of 11^{2401}?

(a) 11 (b) 31 (c) 51 (d) 71

10. The *hotel elevator cubic* polynomial $P(x)$ satisfies $P(11) = 11$, $P(12) = 12$, $P(13) = 14$ and $P(14) = 15$. What is $P(15)$?

(a) 13 (b) 14 (c) 15 (d) 16

11. The common external tangent of two circles has length 2017 and their common internal tangent has length 2009. What is the product of their radii?

(a) 8040 (b) 8044 (c) 8048 (d) 8052

12. Find the sum of all positive rational numbers less than 10 with denominator 30 when written in lowest terms.

(a) 373 (b) 381 (c) 390 (d) 400

13. How many positive integers divide at least two of the numbers 120, 144, and 180?

(a) 11 (b) 13 (c) 15 (d) 17

14. Let $2000 < N < 2100$ be an integer. The last day of year N is a Tuesday and the first day of year $N + 2$ is a Friday. The fourth Sunday of year $N + 3$ is the mth day of January. What is m?

(a) 23 (b) 25 (c) 27 (d) 29

15. In square $ABCD$ with area 1, points A', B', C' and D' are on BC, CD, DA and AB respectively, such that $A'C = B'D = C'A = D'B = 1/n$, for some n. The lines AA', BB', CC' and DD' construct a small square with area 1/1985. Find n.

(a) 28 (b) 30 (c) 32 (d) 35

PART II. Choose the best answer. Each correct answer is worth three points.

1. An arithmetic sequence of positive integers is called *gleaming* if it has at least three terms, exactly one of which is larger than 2017. Find the remainder when the number of gleaming sequences is divided by 1000.

(a) 136 (b) 272 (c) 408 (d) 544

2. Let $\mathcal{S}$ be a set of positive integers such that 1 is in $\mathcal{S}$, and for all integers $n > 1$, an even number of divisors of n are in $\mathcal{S}$. Find the 50th smallest positive integer not in $\mathcal{S}$.

(a) 118 (b) 123 (c) 128 (d) 133

3. Convex pentagon $ABCDE$ satisfies $AB \parallel DE$, $BE \parallel CD$, $BC \parallel AE$, $AB = 30$, $BC = 18$, $CD = 17$, and $DE = 20$. Find its area.

(a) 612 (b) 624 (c) 636 (d) 648

4. The first three terms of an increasing geometric series are x, y, z, all positive integers. Given $1 + \log_2(xy + z) = \log_2(xz + y)$, find the minimum possible value of $x + y + z$.

(a) 50 (b) 55 (c) 60 (d) 65

5. Let $P(n)$ be the product of the non-zero digits of n. Find the largest prime factor of the sum $P(1) + P(2) + \cdots + P(999)$.

(a) 97 (b) 101 (c) 103 (d) 107

6. Find the 50th smallest positive integer whose base two representation has an equal number of zeroes and ones. (For example, the smallest numbers with this property are 10_2, 1001_2, 1010_2, and 1100_2.)

(a) 228 (b) 232 (c) 240 (d) 527

7. Find the greatest positive integer n such that 2^n divides $\operatorname{lcm}\left(1^1, 2^2, 3^3, \ldots, 2016^{2016}\right)$.

(a) 10240 (b) 13440 (c) 13824 (d) 14336

8. Let a, b, and c be the roots of $2x^3 - x^2 + 3x - 2 \cdot 3$. Find the value of $\sqrt{a^2 + 3} \cdot \sqrt{b^2 + 3} \cdot \sqrt{c^2 + 3}$.

(a) 2 (b) 3 (c) 6 (d) 8

9. In triangle ABC, let $AB = 14$, $BC = 15$, and $CA = 13$. Let D be the foot of the altitude from C to AB, M be the midpoint of AB, and N be the midpoint of DM. Find the length of CN.

(a) $\sqrt{145}$ (b) $7\sqrt{3}$ (c) $5\sqrt{6}$ (d) $\sqrt{155}$

10. A strictly increasing arithmetic sequence $a_1, a_2, a_3, \ldots, a_{100}$ of positive integers satisfies

$$a_1 + a_4 + a_9 + \cdots + a_{100} = 1000,$$

where the summation runs over all terms of the form a_{i^2} for $1 \le i \le 10$. Find a_{50}.

(a) 118 (b) 123 (c) 128 (d) 133

PART III. All answers should be in simplest form. Each correct answer is worth six points.

1. Find the value of
$$\frac{1}{0!+1!}+\frac{1}{1!+2!}+\frac{1}{2!+3!}+\frac{1}{3!+4!}+\cdots.$$

2. Find the sum of all positive integers n such that exactly 2% of the numbers in the set $\{1,2,\ldots,n\}$ are perfect squares.

3. Let $\triangle ABC$ be a triangle with $AB = 3$ and $AC = 5$, and suppose there exists a point P on line BC satisfying $AP \cdot BC = BP \cdot AC = CP \cdot AB$. Find the product of all possible values of BC^2.

4. A word is constructed out of five letters A, B, C, D, E and is compressed in the following way: each consecutive string of identical letters is replaced with the length of the string and the letter used. For example, $ABBACCCAA$ is compressed to $1A2B1A3C2A$. Find the expected value of the length of the compressed form of a randomly selected nine-letter word.

5. Let a, b, c, d, e, and f be real numbers. Define the polynomials
$$P(x) = 2x^4 - 26x^3 + ax^2 + bx + c \quad \text{and} \quad Q(x) = 5x^4 - 80x^3 + dx^2 + ex + f.$$
Let S be the set of all complex numbers which are a root of either P or Q (or both). Given that $S = \{1, 2, 3, 4, 5\}$, compute $P(6) \cdot Q(6)$.