VCSMS PRIME

Program for Inducing Mathematical Excellence

Week 5 Homework

Due October 19, 2017

Homework

Due on Thursday, October 19. GMATIC is on Wednesday, school exams on Wednesday and Thursday. Finish your sets early so you have enough time to review. No session on Friday, take a break.

Set A (12) **S1**: Domain and range 1–2. **S5**: Complex numbers 1; Polynomials 2; Polynomial factors 1; Remainder theorem 2–3; Vieta's 1, 4; Coordinate plane 1. **S9**: Coordinate geometry 1–2.

Set B (12) **S1**: Domain and range 3–4. **S5**: Complex numbers 4; Polynomials 4; Polynomial factors 2; Remainder theorem 1; Root-finding 1–2; Vieta's 2–3; Coordinate plane 2–3.

Set C (12) **S5**: Complex numbers 2; Polynomials 5–6; Polynomial factors 4; Root-finding 3–4; Vieta's 5; Coordinate plane 4–7. **S9**: Coordinate geometry 3.

Set D (12) **S5**: Complex numbers 3; Polynomials 2, 7–8; Polynomial factors 5; Root-finding 5–6; Vieta's 6; Coordinate plane 8–10. **S8**: Single-variable extrema 3.

Additional problems

1. The second-degree polynomial $P(x)$ satisfies $P(1) = 1, P(2) = 3, P(3) = 2$. Then $P(P(x)) = x$ has four real solutions, one of which is not an integer. Find the non-integral solution.
2. Let $u > v$ be positive integers, $A(u, v)$, B the reflection of A about $y = x$, C the reflection of B about the y-axis, D the reflection of C about the x-axis, and E the reflection of D about the y-axis. If $[ABCDE] = 451$, find $u + v$.
3. (Resurgence) Complex a, b, c satisfy $|a| = |b| = |c| = 21$ and $ab + bc + ca = 0$. Determine $\left|a^2 + b^2 + c^2\right|$.
4. Find the minimum magnitude of any z satisfying $|z - 4i| + |z - 3| = 5$.
5. Suppose $x^{2017} = 1$ and $x \neq 1$. Evaluate $\dfrac{1}{1+x} + \dfrac{1}{1+x^2} + \cdots + \dfrac{1}{1+x^{2017}}$.
6. Triangle ABC has its circumcenter at the origin, its centroid at $(-1, 13)$ and the midpoint of side BC at $(9, 18)$. Find the maximum possible product of the coordinates of B.
7. (AIME 2005/14) Each of $(0, 12), (10, 9), (8, 0), (-4, 7)$ are on a different side of a square. Find its area.
8. For how many positive integers less than 2017 does $x^2 + x + 1$ divide $x^{2n} + 1 + (x + 1)^{2n}$?
9. Let $f(x) = x^{2016} + 2x^{2015} + 3x^{2014} + \cdots + 2017$ and $z^{2018} = 1$ but $z \neq 1$. Find $f(z)f(z^2)\cdots f(z^{2017})$.
10. (AIME 1992/10) Find the area of the region in the complex plane consisting of all points z such that both $z/40$ and $40/\bar{z}$ have real and imaginary parts between 0 and 1 inclusive.
11. Let $A(0, 0), B(b, 2)$ and $ABCDEF$ be a convex equilateral hexagon with $\angle FAB = 120°$, opposite sides parallel, and y-coordinates from $\{0, 2, 4, 6, 8, 10\}$. Find its area.
12. (AIME 1996/11) Let P be the product of the roots of $z^6 + z^4 + z^3 + z^2 + 1 = 0$ that have a positive imaginary part. Find $\arg P$.
13. Find any geometry problem you haven't solved from the past sets and try to do it with coordinates.