EASY: 2 points each

1 My Pokémon Trainer ID number is $5a3bc$. It is divisible by 45, and all its digits are distinct. Find all possible values of $a+b+c$.

2 Cheating Chester has a way with dice. He never rolls less than a 4, but he also never rolls two 4's in a row. What is the probability that he rolls twice and gets a sum of 10?

3 For what integral values of k is $\dfrac{3k^3+k^2+5k+6}{k+1}$ an integer?

4 The probability that a heart patient survives after surgery in the Sipnayan General Hospital is 90%. Find the probability that, out of five randomly chosen heart patients undergoing surgery, at least four survive.

5 A die with its face numbered 1 to 6 is loaded such that for each face, the probability of it showing that face is inversely proportional to the number on the face. I toss this die and I am told that it is showing a prime. What is the probability that it is showing a 2?

6 A certain function $f(x)$ satisfies the following conditions:

$$\frac{f(x)}{x}+xf\left(\frac{1}{x}\right)=x^2-3x+5$$
$$f(cx)=cf(x), \quad \text{where } c \text{ is a constant}$$

Find $f(4)$.

7 A *piano number* is a number that can be partitioned perfectly into strings of 10101 and 1010101. For example, 10101101010110101 is a piano number: 10101|1010101|10101. How many distinct 40-digit piano numbers are there?

8 If by dialing Niel's phone number 7 492 989, Sarah's finger traveled a total of 2011 mm, how far is the distance from 1 to 2? (numbers are evenly spaced as in a real telephone)

AVERAGE: 3 points each

1 Find the sum of the infinite series

$$1+2\left(\frac{1}{2011}\right)+3\left(\frac{1}{2011}\right)^2+4\left(\frac{1}{2011}\right)^3+\cdots$$

2 Let $a\,?\,b=ab+a+b$ for all integers a and b. Evaluate $1\,?\,(2\,?\,(3\,?\,(4\,?\cdots(2010\,?\,2011)\cdots)))$. You do not have to solve for the exact value of the answer. You may leave an expression.

3 How many positive pairs of integers (a,b) with $a\le b$ satisfy $\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{1}{2011}$?

4 Find the largest integer n such that $2^n \,\Big|\, 2011^{2011^{2011}}+1$.

5 Two circles have radii 30 and 190. If the two external tangents to the circles intersect at 60°, how far apart are the centers of the circles?

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DIFFICULT: 5 points each

1 One day, Russell went to Zheng's house and had a friendly conversation with him. Russell said: "The taxi cab I rode coming here has the plate number 1729." Zheng then replied: "Wow, isn't that great luck?! You got a really special taxi cab." Russell then replied: "Yea, it is the smallest number that can be expressed as the sum of two perfect [cubes in two different ways.]" Zheng then replied: "Yup it is the sum of the cubes of 9 and 10 and it can also be expressed as the sum of the cubes of 1 and 12." Now the question is this: What is the smallest number that can be expressed as the sum of three cubes in two different ways?

2 Suppose x and y are real numbers such that $2x^2 + y^2 - 2xy + 12y + 72 \leq 0$. What is the value of x^2y?

3 In $\triangle ABC$, $\angle ABC$ is obtuse. Point D lies on side $\overline{AC}$ such that $\angle ABD$ is right, and point E lies on side $\overline{AC}$ between A and D such that $\overline{BD}$ bisects $\angle EBC$. Find CE, given that $AC = 35$, $BC = 7$, and $BE = 5$.

4 What is the coefficient of x^{2011} in the expansion of

$$(x+1)^7\left(x^2+1\right)^4\left(x^4+1\right)^5\left(x^8+1\right)\left(x^{16}+1\right)\left(x^{32}+1\right)\left(x^{64}+1\right)\left(x^{128}+1\right)\left(x^{256}+1\right)\left(x^{512}+1\right)\left(x^{1024}+1\right)?$$

5 Let $ABCD$ be a square of side length 26. Let E and F be points on rays $\overrightarrow{AB}$ and $\overrightarrow{AD}$, respectively, so that the area of square $ABCD$ equals the area of $\triangle AEF$. If $\overleftrightarrow{EF}$ intersects $\overleftrightarrow{BC}$ at X and $BX = 12$, determine DF.

VERY DIFFICULT: 8 points each

1 Find the value of

$$\sum_{i=1}^{1006} \left\lfloor \log_2 \frac{2011}{2i-1} \right\rfloor.$$

The $\lfloor x \rfloor$ symbol represents the greatest integer function of x which gives us the greatest integer less than or equal to x.

2 Find the largest integer which is less than $\sqrt{2500} - \sqrt{2501} + \sqrt{2502} - \sqrt{2503} + \cdots - \sqrt{2999} + \sqrt{3000}$.

3 Convex quadrilateral $ABCD$ has right angles $\angle A$ and $\angle C$ and satisfies $AB = BC$ and $AD = CD$. The diagonals $\overline{AC}$ and $\overline{BD}$ intersect at point M. Points P and Q lie on the circumcircle of $\triangle AMB$ and segment $\overline{CD}$, respectively, such that points P, M, Q are collinear. Suppose that $m\angle ABC = 160^\circ$ and $m\angle QMC = 40^\circ$. Find $MP \cdot MQ$, given that $MC = 6$.

SPECIAL ITEM: 12 points

$$WHITE + WATER = PICNIC$$

All of the letters present above represent a distinct integer. Find the numerical value of *PICNIC*.