Sipnayan 2016

Senior High School

November 5, 2016

Very inexact wording for written round. If you have any corrections or additions, please contact me at `cj@cjquines.com`, or through my Facebook account, Carl Joshua Quines.

Written round, two hours

Easy, two points each

1. How many ways are there for 7 people to sit in a row, if two people, Ian and Lemuel, have to sit at least two seats apart? [2400]
2. Find the range of $f(x) = \sqrt{16 \sin^3 x + 65}$. [[7, 9]]
3. How many ways are there to arrange 6 A's and 4 P's, if the string "PPAP" must appear? [42]
4. Given $ab = 7^x, bc = 7^y$ and $ca = 7^z$. Find the value of $\log_7 a + \log_7 b - \log_7 c$ in terms of x, y, and z. $\left[\dfrac{3x - y - z}{2}\right]$
5. Find the last three digits of 35^{13579}. [875]
6. We randomly pick a k-element subset of the set $\{0, 1, 2, \ldots, 50\}$. Find the least value of k such that, no matter how we pick the k numbers, there always exists two even numbers in the subset whose difference is divisible by 13. [39]
7. Find the value of $\log_{2016}\left(\dfrac{1}{2}\right) \log_{2015}\left(\dfrac{1}{3}\right) \cdots \log_3\left(\dfrac{1}{2015}\right) \log_2\left(\dfrac{1}{2016}\right)$. [−1]
8. Two candidates are running for president, each having five supporters. Each supporter only supports one candidate. In the election, each supporter independently has a 50% chance to vote for the candidate they support, or a 50% chance to be lazy and not vote. What is the probability that the election is a tie? (Note that both candidates having no votes is also considered a tie.) $\left[\dfrac{63}{256}\right]$

Average, three points each

1. Given $\dfrac{1}{\sin x} + \dfrac{1}{\cos x} = \dfrac{4}{3}$ and $\dfrac{\sin^2 x}{\cos x} + \dfrac{\cos^2 x}{\sin x} = \dfrac{11}{6}$, find $\sin x + \cos x$. $\left[-\dfrac{1}{2}\right]$
2. If $\log_{a-1}(b+1) = 3$ and $\log_{a+1}(b-1) = 2$, find the value of $(a+1)(b-1)$. [125]
3. Find the sum of the coefficients of a polynomial with degree 4, integral coefficients, and leading coefficient 1, such that both $\sqrt{3}$ and $\sqrt{5}$ are roots. [8]
4. If $20! \cdot 21! \cdots 28! = m \cdot n^3$, where m and n are positive integers such that m is not divisible by the cube of a prime, find m. [825]

5. J, D, and G are circles that are pairwise externally tangent. The centers of J, D, and G form a right triangle. If the radius of D is 8 units and the radius of G is 7 units, then the least value for the radius of J can be expressed in the form $\dfrac{\sqrt{a}+b}{c}$, where a is a positive integer that is not a perfect square, b is an integer and c is a positive integer. Find the value of $a+b+c$. [436]

Difficult, five points each

1. If a and b are positive integers such that $\dfrac{a^3}{b^2} = \dfrac{5}{16}$, find the sum of all possible values of a that are less than 150. [275]

2. Gella's favorite quadratics are all related to the number 20. More specifically, her favorite quadratics are of the form $x^2 + bx + c$, for some integers b and c, such that one root is 20 and the two roots differ by no more than 20. Find the sum of the coefficients of all of Gella's favorite quadratics. [14801]

3. Solve for x: $\sqrt[3]{20x + \sqrt[3]{20x + \sqrt[3]{20x+16}}} = 16$. [204]

4. The roots of $P(x) = x^3 - 5x^2 + 6x - 4$ are x_1, x_2 and x_3. Find the value of $x_1^3 + x_2^3 + x_3^3$. [47]

5. A triangular pyramid is inscribed in a sphere with radius 3 units. If the pyramid has four sides, and all of the sides are equilateral triangles, find the surface area of the pyramid. $\left[24\sqrt{3} \text{ sq. units}\right]$

Very difficult, eight points each

1. The nth floor of Space Invaders hotel has n rooms. The first room in each floor is labeled $100n + 1$, and the labels increment by 1 for each succeeding room. The metal digits used to label each room cost one more than its value: the digit 0 costs \$1, the digit 1 costs \$2, and so on. If Space Invaders hotel has 276 rooms, find the total cost of all the metal digits used to label its hotel rooms. [\$3795]

2. The center of a unit circle is randomly placed in a square of side length 10 units. Find the probability that the circle is contained entirely in the square and does not touch either diagonal.

$$\left[\frac{18 - 8\sqrt{2}}{25}\right]$$

3. There are 2 ways to arrange 2 pairs of parenthesis in a row such that they are matched: these are ()() and (()). Arrangements that are not matched include)(() and ())(. The number of ways to arrange 2016 pairs of parenthesis in a row such that they are matched can be expressed in the form $\dfrac{a!}{b! \cdot c!}$ where a, b and c are nonnegative integers. Find the value of $a+b+c$. [8065]