Sipnayan 2017

Junior High School

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Inexact wording. With thanks to Vincent Dela Cruz and Ralph Daniel Valdres for providing the questions. If you have any corrections, please contact me at `cj@cjquines.com`, or through my Facebook account, Carl Joshua Quines.

Written round, two hours

Easy, two points each

1. Darci has 5 distinct dresses and 8 distinct pairs of shoes. Everyday, she choose a dress and a pair of shoes to wear. Afterwards, she donates it to charity. She then creates a schedule of what she will wear for the next 3 days. How many different schedules can she make? [20160]

2. Kervin makes a mathematical pyramid. He writes one 1 on top, two 2s on the next layer, three 3s on the next, and so on. He stops on the 50th layer, with fifty 50s. What is the sum of his inputs? [42925]

3. In a table, one person can sit on the shorter side and four people on the longer side. If Minnie, Jasmin and Isidro want to sit on the same side of the table with 7 other people, how many ways can this happen? [241920]

4. What is the maximum value of $|\sin x \cos x|$ for $x \in \mathbb{Z}$? $\left[\frac{1}{2}\right]$

5. A circle is tangent to line $2x - y = 4$ at $(4, 4)$. If the center of the circle is on the y-axis, find the equation of the circle. $\left[x^2 + (y-6)^2 = 20\right]$

6. Seven indistinguishable vials are placed in a row. A random vial is poisoned. The vials were "swapped" 13 times, a "swap" being switching the positions of the adjacent vials. What is the probability that the leftmost vial is poisoned? $\left[\frac{1}{7}\right]$

7. Find the sum of all two-digit prime numbers written as $10x + y$ such that $x^2 + 8y = y^2 + 8x$. [152]

8. How many pairs of positive integers (a, b) where $b > a$ exist such that $2017^2 - 2014^2 = b^2 - a^2$? [4]

Average, three points each

1. Let $\theta \in [0, 2\pi]$ such that $\dfrac{\csc^2\theta + \tan\theta}{\cot^2\theta} - \dfrac{\sec^3\theta + \sec\theta}{\csc\theta} = 0$. Find all possible values of θ. $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

2. How many tuples (x, y, z) of positive integers exist such that y and z are multiples of x and $x + y + z = 100$? [200]

3. Find the sum of all positive integers $x < 2048$ such that x is a multiple of 23 and is 6 less than a multiple of 29. [4094]

4. Evaluate $\left(\cos^2 1^\circ + \cos^2 2^\circ + \cdots + \cos^2 90^\circ\right)^2 + \left(\sin^2 1^\circ + \sin^2 2^\circ + \cdots + \sin^2 90^\circ\right)^2$. $\left[\dfrac{8101}{2}\right]$

5. If $f(x) = \log \dfrac{1+x}{1-x}$, find $f\left(\dfrac{3x+x^3}{1+3x^2}\right)$ in terms of $f(x)$. $[3f(x)]$

Difficult, five points each

1. Find all values of x satisfying $\sqrt{x+23-10\sqrt{x-2}} + \sqrt{x+34+12\sqrt{x-2}} = 11$. $[x \in [2, 27]]$

2. Evaluate $\displaystyle\sum_{x=1}^{\infty} \frac{2}{x^3+6x^2+11x+6}$. $\left[\dfrac{1}{6}\right]$

3. A class trial is held with 15 participants. When decided, all 15 participants vote “guilty”, “not guilty”, or “abstain”. When at least half of the votes are guilty, or when this cannot happen, the voting stops. What is the number of possible trials? [2228225]

4. The roots of $2x^2 - 3x + 1 = 0$ are a and b. Evaluate $a^5 + b^5$. $\left[\dfrac{33}{32}\right]$

5. Find the difference of the maximum and minimum solutions in $(0, 2\pi)$ of the equation $4\cos^3 2x - \cos^2 x + 3 = 16\cos^2 x \sin^2 x - \sin^2 x$. $\left[\dfrac{5\pi}{3}\right]$

Very difficult, eight points each

1. The function $f : \mathbb{Z} \to \mathbb{R}$ satisfies $f(a) + f(-a) = -f(a+1)$ for all $a \in \mathbb{Z}$. Given $f(1) = 1$, find the value of $f(2017) - f(2016)$. $\left[\dfrac{3}{2}\right]$

2. Three groups of girl scouts bake 153 boxes of cookies in all. The average number of boxes each girl in group one bakes is 10, the average number of boxes each girl in group two bakes is 7, and the average number of boxes each girl in group three bakes is 6. If there are 21 girls in all three groups, find all possible triples of girls in each group.
$[(3, 15, 3), (4, 11, 6), (5, 7, 9), (6, 3, 12)]$

3. If $f(x) = \dfrac{2017^{2x}}{2017^{2x} - 2017}$, and
$$f(-2016) - f(-2015) + f(-2014) - \cdots + f(0) - \cdots - f(2015) + f(2016) = f(a),$$
for some real number a, find a. [0]

Semifinals A

Easy, two points, thirty seconds

1. A year is a leap year if it is divisible by 4 but not 100 or divisible by 400. How many leap years were there since the Battle of Mactan (1521) until now? [121]

2. The following expression
$$2\sqrt{2}+\cfrac{1}{2\sqrt{2}+\cfrac{1}{2\sqrt{2}+\cdots}}$$
can be simplified into the expression in the form $\sqrt{a}+\sqrt{b}$. What is $a+b$? [5]

3. What is the remainder when 19^{2017} is divided by 7? [5]

4. If $\tan\theta = \frac{1}{3}$, what is the value of $\sin^2\theta + 2\sin\theta\cos\theta + 2\cos^2\theta$? $\left[\frac{5}{2}\right]$

5. Find all ordered pairs (a,b) of positive integers where p,q are prime numbers, such that $p = ab$ and $q = 11a + b$. $[(1,2),(2,1)]$

Average, three points, sixty seconds

1. Let ABC be a triangle with $AC = 7, BC = 24$ and $AB = 25$. Find the area of the circle passing through the midpoint of AC, the midpoint of BC, and point C. $\left[\frac{625}{16}\pi \text{ sq. units}\right]$

2. Given $x + 2y = 9$, find the minimum value of $x^2 - 4x + 4y^2$. $\left[\frac{41}{2}\right]$

3. A greatest integer function is defined to be a function which produces the largest integer less than or equal to x for any real number x. Find all the roots of the greatest integer function $f(x) = \left\lfloor \frac{x^2}{2017} \right\rfloor - 1$. Express your answer in interval notation.
$\left[\left(-\sqrt{4034}, -\sqrt{2017}\right] \cup \left[\sqrt{2017}, \sqrt{4034}\right)\right]$

4. Jasmin thought of a polynomial of degree 2 and dares Isidro to guess what polynomial she has thought of. Isidro first guesses $x^2 + 4x + 1$, but Jasmin says for whatever value of x, that polynomial would be less than what she has thought of. Isidro then guesses $3x^2 + 12x + 9$, but this polynomial would be greater than what she has thought of for any x. Jasmin then gives a hint for her polynomial, if $x = 1$, then the value of the polynomial is 9. What is the polynomial expressed in the form $ax^2 + bx + c$? $\left[\frac{4}{3}x^2 + \frac{16}{3}x + \frac{7}{3}\right]$

5. Evaluate $\sec\left(0.5\cos^{-1}\left(\frac{1}{4}\right)\right)$. Rationalize your answer. $\left[\frac{2\sqrt{10}}{5}\right]$

Difficult, five points, ninety seconds

1. Given $5\sin\theta + 12\cos\theta = 13$, where $\theta \in (0,\pi)$, find $10\csc\theta - 12\sec\theta + 5\cot\theta$. [25]

2. Akira fills an urn with 10 chips such that 1 chip is labeled "1", 2 chips are labeled "2", 3 chips are labeled "3", and 4 chips are labeled "4". She draws 4 chips from the box without replacement. What is the probability the sum of the numbers labeled on the 4 chips is divisible by 3? $\left[\frac{1}{3}\right]$

3. Given $\frac{4x}{y} + \frac{4y}{x} = 9$ and $\frac{4x}{y^2} + \frac{4y}{x^2} = 35$. Find $\frac{4}{x} + \frac{4}{y}$. [28]

4. In rectangle $ABCD$, $AB = 15$ in. and $BC = 12$ in.. F is on CD and G is on the extension of AB such that FG divides BC into two equal segments. If $\angle AFG = \angle CFG$, what is BG? $\left[\dfrac{12}{5}\text{ in.}\right]$

5. Find all ordered triples (p, q, r) where $p < q < r$ such that $p+q+r = -\dfrac{11}{2}$, $p^2+q^2+r^2 = \dfrac{285}{4}$ and $pqr = 70$. $\left[\left(-7, -\dfrac{5}{2}, 4\right)\right]$

Semifinals B

Easy, two points, thirty seconds

1. Given a rectangle whose vertices are located at $M(-5, 6), A(7, -3), T(-5, -3)$ and $H(7, 6)$, find the number of points with integer coordinates located within the interior of rectangle $MATH$. [88]

2. How many integers from 1 to 2017 are not divisible by $2, 3$ or 5? [538]

3. Find the number of possible 7-digit numbers $\overline{abcdefg}$ such that $a < b < c < d < e < f < g$. [36]

4. Minnie downloads a sample test at 30 kbps. After downloading 1/3 of the test, the speed drops to 5 kbps. When half of the remaining were downloaded, the speed increased to 10 kbps. Minnie calculates the average speed of the download. Chang also attempts to get the average speed of the download. However, he only knows the different download speeds, and not the amount downloaded at each speed, so Chang gets the average of the three speeds. Find the difference between Minnie's answer and Chang's answer. [6 kbps]

5. The sum of odd factors of 2,916 can be expressed in the form $2^a + b$, where a and b are positive integers, and a is the greatest number that can satisfy the given conditions. What is $a + b$? [79]

Average, three points, sixty seconds

1. If a and b are positive real numbers such that $a + b = 1$, find the minimum value of $\left(a + \dfrac{1}{a}\right)^2 + \left(b + \dfrac{1}{b}\right)^2$. $\left[\dfrac{25}{2}\right]$

2. Stephanie was drinking a mug of tea with honey. She put 5 teaspoons of honey in her tea, mixed it thoroughly, and drank 3/5 of the mug. She then added 2 teaspoons of honey in her tea, and filled the whole mug with hot water. After mixing the honey in thoroughly, she drinks 1/5 of the mug, then she decided that the tea was not sweet enough. How much honey should be added to the mug in order to make the tea as sweet as it was before she drank the tea?[1] [12]

3. Find the coefficient of x^3 in the expansion of $(1-x) + (1-x)^2 + (1-x)^3 + \cdots + (1-x)^{10}$. [−330]

[1] Replaced after D5. What is the sum of the positive integers less than 100 that cannot be expressed as the sum of two or more consecutive positive integers? Answer: 127.

4. Mabel decided to buy a jar of Surprise Me! Candies. In every jar there is a guarantee that 2/3 of the candies are sweet while the remaining are sour. After Mabel ate 10 of the candies, the probability of getting a sweet candy is now 3/4. What is the minimum amount of sweet candies in the jar at the beginning for this to be possible? [12]

5. Phineas decided to eat all the cupcakes and muffins in the Tristate Area so that Buford would not be able to buy any. In order to eat that many pastries, he decided to clone himself. Phineas and all his clones have the same appetite. In one hour, 100 clones can eat 1500 cupcakes and 1000 muffins. In two hours, 60 clones can eat 1200 cupcakes and 1500 muffins. In three hours, 50 clones can eat 750 cupcakes and x muffins. Find x. [2250]

Difficult, five points, ninety seconds

1. Three unit circles are externally tangent to each other and internally tangent to a larger circle A. Similarly, six unit circles are externally tangent to each other and internally tangent to a larger circle B. What is the ratio of the radius of circle A to the radius of circle B? $\left[\frac{1}{3}+\frac{2\sqrt{3}}{9}\right]$

2. Find the value of $\frac{1}{6}-\frac{3}{6^2}+\frac{6}{6^3}-\frac{10}{6^4}+\frac{15}{6^4}-\cdots$. $\left[\frac{36}{343}\right]$

3. Suppose that the roots of $x^3-3x^2-3x+7=0$ are a, b and c, and the roots of $x^3+rx^2+sx+t=0$ are $a+b$, $b+c$ and $c+a$. Find the value of t. [2]

4. A number is a *werpa* number if it has at least two of the same digits and the sum of its digits is even. For example, 969 is a *werpa* number but 995 is not. What is the sum of the three-digit numbers that are not *werpa* numbers? [422,620]

5. Thor's lightning blast follows the structure of the graph $y=\sin x$. Prince Zuko's lightning bending follows the structure of the graph $y=2\cos\left(x+\frac{5}{2}\pi\right)$. On the other hand, Pikachu's Thunder follows the graph of $y=7\sin(x+4\pi)$. The three of them were standing on a cliff 200m above the sea. If x is the horizontal distance and y is the vertical distance, how far away are their lightning attacks from the point of origin when all three lightning attacks intersect each other for the 2017th time? Give your answer in meters and you may express it in terms of π. [2017π m]

Finals

Teams sequentially choose question order, and can skip between rounds. If not stated, mechanics are as usual.

Spongebob (SS) groups teams into three-three: two teams getting correct get 1.5 times the points, one team getting correct gets twice the points.

Gravity Falls (GF) gives teams the choice to get twice the alloted points if correct and deducted the number of points if wrong or blank.

Fairly Odd Parents (FOP) gives team the choice to wish. Wishing gives twice points if correct, penalized next FOP question if wrong. If penalized, a team cannot wish and gets half points if correct. Teams cannot wish in last FOP question.

Jimmy Neutron (JN) gives teams a flag to raise: raising before 1/3 of the time is up gives twice if correct, deduct if wrong, none if blank; raising before 1/2 the time is up gives 1.5 times if correct, deduct 0.5 if wrong, none if blank. Answers cannot be changed after raising the flag.

Phineas and Ferb (PF) gives teams the choice to allow a randomly chosen team member to sit out for 2.5 times the points if correct.

Easy, 2 points, 30 seconds

SS. The sum of 2160 consecutive positive integers is a perfect cube. What is the minimum value of the largest of these integers? [3222]

GF. Find all possible values of x where $x^{\log x} = 1000000x$. $[1000, 0.01]$

FOP. Whilce draws an equilateral triangle with side length 20 cm. He wants to change the length of its base so that its area is still preserved and that the lengths of the other two legs remain the same. What should the length of the new side be? $\left[20\sqrt{3}\text{ cm}\right]$

JN. Seven people are seated in a round table. They decided to play a game that starts with a person flipping a fair coin. If the result is heads, the person will pass the coin to the person on his left. If the result is tails, the person will pass the coin to the person to the right. The first person who is able to flip the coin twice wins. If Engel starts the game, what is the probability of him winning the game? $\left[\frac{33}{64}\right]$

PF. Find the sum of all possible values of x satisfying $9x^2 - 24x - \frac{24}{x} + \frac{9}{x^2} = 290$. $\left[\frac{8}{3}\right]$

Average, 3 points, 60 seconds

SS. What is the sum of the positive integers n where $1 \le n \le 200$ such that $\frac{n^2+8}{n^2+10}$ is a fraction in lowest terms? [10000]

GF. Let an equilateral triangle and a circle be drawn such that the base of the triangle is tangent to the circle (in diagram: circle passes through opposite vertex). What is the ratio of the area of the circle with the area of the triangle? $\left[\pi\sqrt{3} : 4\right]$

FOP. Square $FERB$ has side length 17, and points P and I are exterior to the square such that $EP = BI = 8$ and $FP = RI = 15$, find PI^2. [1058]

JN. Find $\log_3 x$ if $\log_3(\log_{27} x) = \log_{27}(\log_3 x)$. $\left[3\sqrt{3}\right]$

PF. Evaluate $\frac{1}{2017} + \frac{2}{2017^2} + \frac{3}{2017^3} + \cdots$. Express your answer in the form $\frac{a}{b^2}$. $\left[\frac{2017}{2016^2}\right]$

Difficult, 5 points, 90 seconds

SS. Find all ordered pairs of positive integers (x, y) such that x divides $y+1$ and y divides x^2+1. $[(1,1),(1,2),(2,1),(2,5),(3,2),(3,5)]$

GF. If a and b are positive real numbers such that $\cos(x+2y) = a$ and $\cos(x+3y) = b$, what is the maximum value of $\sin(2x+5y)$ in terms of a and b? $\left[a\sqrt{1-b^2} + b\sqrt{1-a^2}\right]$

FOP. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x+1) + f(y-1) = f\left(\frac{x+y}{2}\right)$. Evaluate $\left(\frac{f(1)+f(99)}{99!} - 1\right) + \left(\frac{f(2)+f(98)}{98!} - 2\right) + \cdots + \left(\frac{f(98)+f(2)}{2!} - 98\right) + \left(\frac{f(99)+f(1)}{1!} - 99\right)$. [−4950]

JN. Find the minimum value of $\sqrt{(x+3)^2+49} + \sqrt{(x-5)^2+64}$. [17]

PF. Find all ordered triples (a, b, c) satisfying the following equations:

$$\ln(5ab) = \ln(a^{\ln b})$$
$$\ln(5ac) = \ln(a^{\ln c})$$
$$\ln(bc) = \ln(b^{\ln c}).$$

$\left[(5e^2, e^2, e^2), \left(\frac{1}{5}, 1, 1\right)\right]$

Very Difficult, 8 points, 120 seconds

SS. Point $A(x, y)$ is moving along the line $x + 3y = 6$, while point $B(u, v)$ is moving along the line $4u + v = 10$. What is the absolute value of the difference of the minimum values of $3^x + 27^y$ and $16^u + 2^v$? [10]

GF. A circular coin of radius 1 cm is thrown onto a regular hexagonal grid, or in other words, a grid whose cells are regular hexagons. If the coin has a 25% chance of landing entirely inside one of the cells, what is the side length of a cell? $\left[\frac{4\sqrt{3}}{3} \text{ cm}\right]$

FOP. A nonnegative number is said to be *n-riffic* if its base n representation consists only of 1s and 0s. Let $L(n)$ denote the number of n-riffic numbers less than 2017. Find $L(2) + L(3) + L(5) + L(7) + L(11)$. [2209]

JN. Consider the white keys of the piano. The white keys follow the scale ABCDEFG, after which it loops back to A; similarly walking backwards from A leads back to G, then F, then E, and so on. Suppose a giant ant is placed on the A key in the middle of the keyboard. The ant then takes 10 steps: a step moves the ant from the key it is currently on to either one of the keys adjacent to it (chosen randomly with equal probability). Suppose that the ant always has ample space to move either left or right. What is the probability that after 10 steps, the ant ends on any A key? Express the answer as a fraction in simplest terms. $\left[\frac{63}{256}\right]$

PF. Consider triangle JMC with $JM = 3$, $MC = 4$ and $CJ = 5$. Find the value of $\frac{\sin J + \sin M}{\cos J + \cos M} + \frac{\sin M + \sin C}{\cos M + \cos C} + \frac{\sin C + \sin J}{\cos C + \cos J}$. [6]