Sipnayan 2017

Senior High School

November 11, 2017

Inexact wording for written round, but otherwise accurate. Thanks to John Patrick Bas for providing oral round questions. If you have any corrections, please contact me at `cj@cjquines.com`, or through my Facebook account, Carl Joshua Quines.

Written round, two hours

Easy, two points each

1. Find the number of positive integers n such that $(n+1)^n + n^{n+1}$ is divisible by n. [1]

2. Sheen chooses 3 cells at random on a 4×4 grid. What is the probability that the three squares are horizontal or vertical, without any gaps in between? $\left[\dfrac{1}{35}\right]$

3. Let a circle have center O and radius 8. Points A, B, C, D are on circle O such that BC is a diameter, AD is perpendicular to BC. Let E be the intersection of BC and AD. It is known that $\sin \angle DAB = \dfrac{1}{4}$. What is the length of AE? $[\sqrt{15}]$

4. Phineas, Ferb, Isabella, Baljeet, and Buford play a game. Phineas starts with the number 32, and then Ferb says the number that is 50% more than 32. Then Isabella says the number that is 50% more than what Ferb said, and Baljeet and Buford do the similar. Then Phineas says the number that is 50% more than Buford's, and the cycle continues. The game ends right before someone says a number that isn't an integer. What is the sum of all the numbers said? [665]

5. What is the sum of the coefficients of the even-degree terms of $\left(x^3 + \dfrac{2}{x}\right)^4$? [81]

6. Bimb has an average of 94.2 in Math, English, Spelling, History, and Science. He has a grade of 100 and 92 in Spelling and History, respectively. His average grade in English and Spelling is greater than his average grade in Math and Science by 2.5. The average of his non-math grades is 94.5. If Bimb's grade in Math was 3 points higher, and Bimb's grade in Science was 2 points lower, what would be his average in English and Math? [94]

7. Find the sum of the digits of $\dfrac{10^{20} - 2^{11} \cdot 5^{10} + 1}{10^2 - 2^2 \cdot 5^1 + 1}$. [82]

8. Inez has a set of six refrigerator magnets in the shape of the letters A, C, D, G, O, and T. She is amused that she can spell the names of her favorite animals, CAT, DOG, $GOAT$, and $TOAD$. How many arrangements of the six letters contain at least one of her favorite animals as a substring? (For example, **DOG**ACT contains DOG, but **DO**A**G**CT does not.) [58]

Average, three points each

1. Find the minimum of $\frac{1}{x^{2017}} + \frac{1}{x^{2016}} + \cdots + \frac{1}{x} + x + x^2 + \cdots + x^{2017}$ over all positive real numbers x. [4034]

2. The Nickelodeon Math Club 30 students, some from Retroville and the remainder from Bikini Bottom. Each pair of students solves a problem over a year. Working together, two Retroville students take an hour, a Retroville student and a Bikini Bottom student take 4 hours, and two Bikini Bottom students take 3 hours. If the sum of all the hours spent solving is 1425, find the number of Retroville students. [15]

3. Wanda shows a trick to her audience using the sequence $5, 6, 7, 8, 3, 2, 1$, called "Righteous Minimum Query!" She asks the audience to pick a substring of the sequence, like 5678, 7832, or 7, and she gives the smallest number in the substring. What is the sum of Wanda's answers for all possible substrings of the sequence? [94]

4. Carl and Carly play a game called "Chocolate NIM". They have a jar with 10 Flat Tops. In each turn, they take a positive integer number of Flat Tops out of the box, and Carl always goes first. The game ends when the jar is empty. How many possible games of Chocolate NIM are there? (Two games are different if they have a different number of turns, or if there exists a turn i such that the number of Flat Tops taken in that turn differs in either game.) [512]

5. Triangle ABC has cevians BD and CD intersecting at F. Given $\frac{AE}{EB} = \frac{2}{5}$ and $\frac{AD}{DC} = \frac{4}{3}$, find $\frac{DF}{FB} + \frac{EF}{FC}$. $\left[\frac{118}{105}\right]$

Difficult, five points each

1. The sequence $\{a_n\}$ satisfies $a_0 = 1669$, $a_1 = 2017$, and $a_n = 2a_{n-1} + a_{n-2}$ for all $n \geq 2$. Evaluate $\sum_{i=1}^{\infty} \frac{a_{i-1}}{a_i^2 - a_{i-1}^2}$. $\left[\frac{1}{696}\right]$

2. Mable and Dipper Pines each have a bag containing 9 marbles, numbered from 1 through 9. They each remove one marble and add the numbers of the remaining 8 marbles. Find the probability that Mabel's sum and Dipper's sum differ by a multiple of 4. $\left[\frac{7}{27}\right]$

3. Square $ABCD$ has side AB on the line $y = x + 4$ and points C and D on the parabola $x = y^2$. Find the sum of all possible areas of $ABCD$. [68]

4. A trapezoid is inscribed in a circle of radius 2, such that one side coincides with a diameter. Find the maximum possible area of the trapezoid. $[3\sqrt{3}]$

5. A series of 12 lights are in a row. In a single turn, a light can be turned on if and only if it is adjacent to a light that is previously on. Initially, the 1st, 6th, and 12th lights are on. In how many possible sequences can all the lights be turned on? (Two sequences are different if there exists a turn i in which the light turned on differs in either sequence.) [16128]

Very difficult, eight points each

1. Positive real numbers a, b, c, d satisfy $a^2 + b^2 - \frac{1}{2}ab = c^2 + d^2 + \frac{1}{2}cd = 256$ and $ac + bd = 240$. Determine the maximum value of $(ab + cd)^2$.[1] [61440]

2. Find the sum of the first 2017 terms of the sequence $1, 1, 2, 1, 2, 3, 4, 1, 2, 3, 4, 5, 6, 7, 8, \ldots$ [669789]

3. Find all triples (a, b, c) of positive integers such that $a! = 4 \cdot b! + 10 \cdot c!$. $[(4, 1, 2), (6, 5, 4), (14, 13, 13)]$

Semifinals A

Easy, two points, thirty seconds

1. Mr. Krabs is already bankrupt, so he bought a magic money box in Rock Bottom. The box has two buttons – a blue button and a white button. Initially, the box has \$0. When Mr. Krabs presses the blue button, \$1,000 is magically added to the box. On the other hand, when he presses the white one, the amount inside the box doubles. If Mr. Krabs wants to earn exactly \$143,000, at least how many times should he push the blue button in the box? [5]

2. Vincent mentioned his favorite two-digit number at work one day; to his surprise, Angelo's two-digit number was the same as *his* number, only the digits were reversed! Furthermore, Angelo pointed out that if they concatenated their numbers, the resulting four-digit number would be a perfect cube! What is the four-digit number? [1331]

3. Find the last two digits of $4^{2017} + 7^{2017}$. [91]

4. If $\displaystyle\sum_{n=0}^{\infty} (\sin x)^{2n} = 8$, what is $\cos(2x)$? $\left[-\frac{3}{4}\right]$

5. Aunt Patricia recalled a superstition that she heard a long time ago during her time in the Sharp Corporation that goes like this: Draw two cards from a standard deck of 52 playing cards. If the drawn cards are the King and Queen of the same suit, then you and your crush will be destined for eternal happiness! However, if you draw a King and Queen of different suits, you will be cursed with unrequited love! Otherwise, nothing particularly interesting happens. According to the superstition, what is the probability she will be cursed? Express your answer as a fraction in lowest terms. $\left[\frac{2}{221}\right]$

Average, three points, sixty seconds

1. Let $x, y \in \left[0, \frac{\pi}{2}\right]$. If $\sin x = \frac{5}{13}$, $\sin y = \frac{15}{17}$, what is the value of $\tan(x + y)$? $\left[\frac{220}{21}\right]$

[1] Since this is the only problem I didn't get in theory: a, b, c, d form a cyclic quadrilateral in order due to law of cosines, diagonal separating a, b and c, d has length 16 and by Ptolemy's, the other diagonal is 15. The area is $\frac{1}{2}ab\sin\alpha + \frac{1}{2}cd\sin(180-\alpha) = \frac{1}{2}(ab+cd)\sin\alpha$, so it remains to maximize the area, which is when the diagonals are perpendicular. Perpendicular diagonals make the area 120, and from the cosine law $\cos\alpha = \frac{1}{4}$ so $\sin\alpha = \frac{\sqrt{15}}{4}$. This means $ab + cd = 64\sqrt{15}$, and its square is 61440.

2. A *pseudo-binary number* is a positive integer whose decimal representations is composed of only zeroes (0s) and ones (1s). What is the smallest *pseudo-binary number* divisible by 13? [1001]

3. The Sharp Mathematician Kenny has red, blue, green, and yellow socks in his drawer. Suppose Kenny's morning routine is that he reaches into his drawer and pulls out a random sock (any of the four colors can be chosen with equal probability) one by one until he has two socks of the same color. How many distinct possible morning routines exist? Two routines are considered different if and only if the number of total socks pulled out is different OR there exists a number i such that the ith sock pulled out is a different color in the two routines. [196]

4. Jughead is given a segment of length N. Then, he generates two random real numbers in the range $[0, 2N]$ (with uniform distribution) N_1 and N_2. He makes two more segments such that one's length is equal to N_1 and the other's length is equal to N_2. What is the probability that the three segments can be connected vertex to vertex to form a triangle? Express your answer as a fraction in lowest terms. $\left[\frac{5}{8}\right]$

5. How many ordered pairs of integers (m, n) where $m, n \in [-2017, 2017]$ are there such that $x^3 + y^3 = m + 3nxy$ has infinitely many integer solutions (x, y)? [25]

Difficult, five points, ninety seconds

1. Gordon Ramsay has a plate in the design of a 9×9 chessboard. He has 9 pieces of *foie gras*, and he wants to plate them in such a way that one piece of *foie gras* takes up the space of one 1×1 square, the corner squares do not have any *foie gras*, and each row and column only has one piece of *foie gras* each. In how many ways can he do this? Give the exact value. [211680]

2. Solve for x in $\log_x 36 + 6\log_{36} x = 5$. $\left[6, \sqrt[3]{36}\right]$

3. A nonnegative function $f(x)$ satisfies $f(x)f(y) = xf(y) + yf(x) + x + y + 1$. What is $f(x^2)$? $\left[2x^2 + 1\right]$

4. Simplify the following expression in terms of $\sin x$ and $\cos x$:

$$\frac{\sin x(\sin^2 x + \cos^2 x) + \csc x \tan x}{(\csc x + \cot x)(\csc x - \cot x)\sec x} + \cos x\left(\frac{\sin^2 2x}{2} + \sin^4 x + \cos^4 x\right).$$

$[\sin x + \cos x]$

5. Let a and b be the roots of $5x^2 + \left(\sqrt{5} - 5\right)x + \sqrt{30 + 10\sqrt{5}} = 0$. Find $a^2b + ab^2 - a - b + ab - 1$. $\left[\frac{2\sqrt{5} - 1}{5}\right]$

Semifinals B

Easy, two points, thirty seconds

1. Find the value of $x > 0$ in $x^{x^{x^{\cdots}}} = 4$.[2] $\left[\sqrt{2}\right]$

[2] Actually wrong, should be no solution. Without resorting to regions of convergence bla bla, consider the equation $x^{x^{x^{\cdots}}} = 2$, which by the same method gives the answer $\sqrt{2}$. It's impossible for it to equal both, and in this case it equals 2 rather than 4 if you type out the first few on a calculator.

2. Isabella decided to surprise Phineas by writing a three-digit positive integer on the whiteboard. Unfortunately, Phineas accidentally smudged the middle digit before he could see it. Isabella however noticed that the remaining two digits coincidentally formed the square root of her original number! What is the sum of all possible values for Isabella's original three-digit number? [221]

3. Spongebob, Patrick, and Mr. Krabs are shopping together. Spongebob sees some Sharp calculators and decideds to buy 5 calculators for himself with their shared money. Not to be outdone, Patrick buys 9 calculators for himself with their shared money. Mr. Krabs gets angry and then buys 13 calculators for himself with their shared money. This continues, with Spongebob buying 17 calculators, and so on. However, the three of them only have enough money for at most 500 calculators total. How many calculators in total did the three end up buying? [495]

4. The quadratic function $U(x)$ has $U(1) = 6, U(2) = 4U(1) + 4, U(3) = 2U(2)$. What are the zeroes of the function? $\left[\frac{2}{3}, -5\right]$

5. AD is the angle bisector of triangle ABC. If $AB = 5$, $DC = 8$, and $BC = 12$, what is the area of an equilateral triangle that has the same perimeter as triangle ABC? $\left[\frac{81\sqrt{3}}{4}\right]$

Average, three points, sixty seconds

1. Find the largest prime number that divides $1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + 2014 \times 2015 \times 2016$. [2017]

2. Find the last two digits of 3^{2017}. [63]

3. Earl fills an urn with 15 chips such that k chips are labeled $\frac{1}{k}$ for $k = 1, 2, 3, 4, 5$. Then, Marie takes 5 chips from the Earl's urn without replacement. What is the probability that the sum of the numbers labeled on the chips is greater than 1? $\left[\frac{3002}{3003}\right]$

4. Spongebob wants to treat his friends Patrick and Sandy to some Krabby Patties. However, he wants to twist things up. He asks each friend to choose a number such that the number of Krabby Patties that each friend receives is a multiple of that number. Patrick then chooses 7 and Sandy chooses 6. Before he heated up the grill, he determined the maximum number of Krabby Patties he should cook such that however they will be distributed (subject to the conditions), he's guaranteed to have some left for himself. After cooking the first batch, Sandy then decides to change her number to 8 because of her belief in lucky numbers. How many more Krabby Patties should Spongebob cook such that his conditions are still met? [12]

5. Carl has a collection of 100 marbles, labeled $1, 2, 3$, until 100. He then chooses two random marbles and multiplies the values written on them. What is the probability that the product of the numbers is a perfect cube and is a number in his marble collection? $\left[\frac{7}{4950}\right]$

Difficult, five points, ninety seconds

1. The value of $\lim_{x \to 1} \frac{1-\sqrt[3]{x}}{1-\sqrt{x}}$ can be expressed as a fraction $\frac{a}{b}$, where a and b are relatively prime integers. What is the value of ab? [6]

2. Solve for x: $\log_{16}(\log_4 x) = \log_4 \log_{16} x$. [256]

3. Find all values of the constant k such that $5^{25x} + 5^{24x} + \cdots + 5^{2x} + 5^{x} = k - 5^{-x} - 5^{-2x} - \cdots - 5^{-24x} - 5^{-25x}$ has at least one real solution. [$k \geq 50$]

4. Let f be a function satisfying $f(5x - 3y) = 2f(x)f(y)$, for all $x, y \in \mathbb{R}$, and $f(0) \neq 0$. Evaluate $\frac{f(2017) + f(2015) + \cdots + f(3) + f(1)}{f(2016) + f(2014) + \cdots + f(4) + f(2)}$. $\left[\frac{1009}{1008}\right]$

5. What is the remainder when $3^{5^{7^{11}}}$ is divided by 13? [9]

Finals

Teams sequentially choose question order, and can skip between rounds. If not stated, mechanics are as usual.

Spongebob (SS) groups teams into three-three: two teams getting correct get 1.5 times the points, one team getting correct gets twice the points.

Gravity Falls (GF) gives teams the choice to get twice the alloted points if correct and deducted the number of points if wrong or blank.

Fairly Odd Parents (FOP) gives team the choice to wish. Wishing gives twice points if correct, penalized next FOP question if wrong. If penalized, a team cannot wish and gets half points if correct. Teams cannot wish in last FOP question.

Jimmy Neutron (JN) gives teams a flag to raise: raising before 1/3 of the time is up gives twice if correct, deduct if wrong, none if blank; raising before 1/2 the time is up gives 1.5 times if correct, deduct 0.5 if wrong, none if blank. Answers cannot be changed after raising the flag.

Phineas and Ferb (PF) gives teams the choice to allow a randomly chosen team member to sit out for 2.5 times the points if correct.

Easy, 2 points, 30 seconds

SS. Find $\sin 2x \cdot \tan 2x$ if $\cos x = \frac{4}{5}$. $\left[\frac{576}{175}\right]$

GF. For how many positive integers n is $\log(\log(\log n))$ is defined, but $\log(\log(\log(\log n)))$ is not? Note that $\log x = \log_{10} x$. [9,999,999,990]

FOP. A star is placed on the bottom-left corner square of a 3×3 grid. To reach its full potential, it must be on the top-right corner of the grid. A fair coin is flipped repeatedly. Each time that the coin shows heads, the star is moved one space upwards; each time the coin shows tails, the star is moved one space to the right. Assuming that the star may move off the grid, what is the probability that the star will reach its full potential? $\left[\frac{3}{8}\right]$

JN. A number is called *lodi* if it has at least 4 positive factors. Consider the sum of distinct *lodi* coefficients of $(x+y)^7$. How many positive factors does it have? [8]

PF. Josep is at a math competition. The question given in the do-or-die question was "The probability of winning Sipnayan is $\frac{3^2+2^3}{5!}$. What is the probability of not winning Sipnayan?" In his panic, Josep has dropped his glasses before he read the question slip, so he read the probability as $\frac{3^3+2^2}{51}$. The product of the correct answer and Josep's answer (assuming Josep solved using a correct procedure, with only the given as his mistake) can be expressed as a fraction in lowest terms $\frac{a}{b}$. What is $a+b$? [409]

Average, 3 points, 60 seconds

SS. Find x if $\log\left(\frac{2x}{3}\right)+\log\left(\frac{9}{4x^2}\right)+\log\left(\frac{8x^3}{27}\right)+\cdots+\log\left(\frac{3^{2016}}{2^{2016}x^{2016}}\right)+\log\left(\frac{2^{2017}x^{2017}}{3^{2017}}\right)=$ 2018. (Note that $\log x=\log_{10} x$.) [150]

GF. A 5-digit number is made using each of the digits $3, 6, 7, 8$, and 9 exactly once. An *orbskie* is a number that is divisible by 4 but not by 8. What is the probability that a number formed with the digits is an *orbskie*? $\left[\frac{1}{15}\right]$

FOP. Timmy Turner's room has been filled with beads because of a wish gone wrong. Cosmo, Wanda, and Poof try to empty the room by throwing beads out of the window. Poof starts by throwing out 1 bead. Wanda then throws out 2 beads, then Cosmo follows with 4. This continues, with the next person throwing out twice as many beads as the previous person. It takes them each 4 seconds per turn to throw out the beads. If it takes them 1 minute to throw all the beads out, how many beads did Wanda throw out in total? [9362]

JN. Given a, b, c, x, d are positive integers and $\overline{abc}_x=d_{10}$ and $b^2=4a(c-d)$. Find all possible values of x. [none]

PF. If $x>4$, what is the minimum value of $\frac{x^4}{(x-4)^2}$? [256]

Difficult, 5 points, 90 seconds

SS. Merry Christmas in November! Mr. Sharp has a traditional Math Parol which can be constructed in the following manner: consider a regular hexagon with six vertices; draw an extra vertex in the center, and draw edges connecting it to each of the original vertices. Suppose he wants to place LEDs on the vertices, and he has four colors: Red, Blue, Green, and Yellow. How many ways can the seven vertices be assigned LEDs such that no two adjacent vertices have the same color? Two vertices are considered adjacent if they share an edge. Note that the Parol is to be hung upright, so rotations and reflections of the hexagon are considered distinct from each other. [264]

GF. In cyclic quadrilateral $ABCD$, $AB \cong AD$. If $AC=10$ and $\frac{AB}{BD}=\frac{5}{6}$, find the maximum possible area of $ABCD$. [48]

FOP. Find the value of $\tan\left(\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{2}{5}\right)+\tan^{-1}\left(\frac{3}{5}\right)+\tan^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{5}{5}\right)\right)$. $\left[-\frac{387}{613}\right]$

JN. A "petmalu jeep" makes 2018 stops in one trip. At the start of the trip, b people are in the jeep. Let $a_i \neq 0$ be the number of people in the "petmalu jeep" just before the ith stop. At the ith stop, if a_i is even, $\frac{a_i}{2} - 1$ people leave the jeep. Otherwise, if a_i is odd, then $2a_i + 1$ people enter the jeep. Find the minimum value of b such that only 3 people are left after the 2017th stop. [4]

PF. The Grand Geometric Explorer Louie Sharp is recording his 6-Day exploration through the Tetra Wilderness, a Tetrahedron World where each vertex is labeled A, B, C and D. Louie begins on Vertex C (but he does not put this in his journal). Every day, Louie travels to a vertex adjacent to the one he is currently on, and logs the name of that vertex into his journal for the day (Example: Day 1, visited Vertex D). However, Louie wants the entry for the 6th and final day to be on the same vertex he started on initially. How many possible journeys through the Tetra Wilderness are there? [183]

Very Difficult, 8 points, 120 seconds

SS. Find the smallest positive integer x such that

$$2\left[\cos\left(\frac{\pi}{2017}\right)\sin\left(\frac{\pi}{2017}\right) + \cos\left(\frac{4\pi}{2017}\right)\sin\left(\frac{2\pi}{2017}\right) + \cdots + \cos\left(\frac{x^2\pi}{2017}\right)\sin\left(\frac{x\pi}{2017}\right)\right]$$

is an integer. [2016]

GF. A certain credit card has the shape of a rectangle of dimensions 3 units $\times$ 4 units. If you rotate the card about one of its diagonals, what is the volume of the resulting solid of revolution? (as shown in the diagram below) $\left[\frac{4269\pi}{320}\right]$

FOP. Lake Nick is an enormous underwater lake enclosing 2017 islands arranged at the vertices of a regular 2017-gon. Adjacent islands are joined with exactly two bridges. One day, Spongebob took a wrong bus from Bikini Bottom and ended up in one island in Lake Nick. When Spongebob remembered that the Hash Slinging Slasher was living in the lake, he started to think of destroying all the bridges. If the island he is on has at least one bridge still joined to it, he randomly selects one such bridge, crosses it, and immediately destroys it. Otherwise, he stops. What is the probability that Spongebob destroys all the bridges before he stops? Express your answer in the form $a\left(\frac{b}{c}\right)^d$, where $a, b, c, d \in \mathbb{Z}^+$ and b, c are prime numbers.

$\left[1009\left(\frac{2}{3}\right)^{2016}\right]$

JN. Sipnayan is now on its 19th year and based on statistics, it has been found that the number of participants each year is determined by the function $f(x) = 5f(x-1) - 6f(x-2)$. If the number of participants in the first and second year were 210 and 450 respectively, how many participants did Sipnayan have this year? Express your answer in the form $a \times b^c + d \times e^f$. $\left[90 \times 2^{19} + 10 \times 3^{19}\right]$

PF. The diameter of the circumcircle of $\triangle ABC$ is 25. The lengths of AB, BC, CA are all positive integers with $AB > BC$. The distances from the circumcenter O to the sides AB, BC are also integers. Find $AB + BC + CA$. [42]