Sipnayan 2018

Senior High School

December 1, 2018

Thanks to Dan Baterisna and Andres Gonzales for the written round questions; and to Nathanael Joshua Balete for the last figure. Inexact wording for the written round. If you have any corrections or additions, please contact me at `cj@cjquines.com`.

Written round, two hours

Easy, two points each

1. Peter Parker rolls five fair six-sided dice and gets the maximum of the five values that appear. He thinks he is unlucky, because the maximum of the rolls was only 2. What is the probability this happens? Express the numerator and denominator as products of prime powers. $\left[\dfrac{31}{2^5 \cdot 3^5}\right]$

2. Suppose for some real numbers a and b, $a^3 + b^3 = ab - a^2 - b^2$, and $-ab = |a|\,|b|$. Determine the set of all possible values of $a + b$. $[\{-1, 0\}]$

3. Let $A = \sqrt{93 + 28\sqrt{11}}$. Find the value of $\lfloor A \rfloor + \{A\}^2$, where $\lfloor A \rfloor$ is the integer part of A and $\{A\}$ is the fractional part of A. $\left[93 - 24\sqrt{11}\right]$

4. Captain America was surprised to note that both $7 + 1$ and $1 + 7$ evaluate to 8. How many different arithmetic expressions equal to 8 are there which use only the addition operation and *at least* two integers, all of which are positive? Note that order matters, i.e. $1 + 6 + 1$ and $1 + 1 + 6$ and $6 + 1 + 1$, for instance, are all counted as different expressions. [127]

5. How many n in the inclusive range $[1, 2018]$ can be expressed as the sum of two composite numbers? [2010]

6. Let $n \in \mathbb{Z}^+$ such that both n and $n + 55$ are perfect squares. Give all possible values of n. $[9, 729]$

7. Let DC and DE be chords of the same circle. Let EF be a chord parallel to DC, and let CH be a chord parallel to DE. Finally, let G be the point on the arc EC (the arc which does not contain point D) such that $EG : GC = 2 : 3$. If $m\angle CDE = 40^\circ$, find $m\angle FGH$. $[80^\circ]$

8. Evaluate: $\sec^2 75^\circ + \csc^2 75^\circ$. [16]

Average, three points each

1. Find the largest divisor of $6\,006\,006\,006$ that does not exceed $60\,000$. $[59\,406]$

2. Let F be the set of unique factorizations of the number $10\,010$. More precisely, a unique factorization is an unordered multiset (an element can appear more than once) of integers greater than 1 such that the product of all its elements is $10\,010$, for example $\{10, 1001\}$ could be one such element of F. Note that by this definition, $\{1001, 10\}$ is equivalent to $\{10, 1001\}$. How many elements are there in F? [52]

3. In this coming 2018–19 NBA, the owner of the Golden State Warriors had a peculiar idea which was having the seats in a row be painted in yellow or blue, in a way such that the number of consecutive seats painted in the same color is always odd. How many ways can this be done with a row of 14 seats? A consecutive block can also consist of just one seat. [754]

4. How many ordered triples (a, b, c) of positive integers are there such that $abc = 64\,800$? [1890]

5. Given that $\log_{10} 2 \approx 0.30103$, find the smallest integer n such that 4^n has 100 digits when fully expanded. [165]

Difficult, five points each

1. For an integer n, let $\sigma(n)$ be the sum of all the positive divisors of n. For how many integers n in the inclusive range $[1, 500]$ will $\sigma(n)$ be a prime number? [7]

2. Let $P = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{10000}}$. Find $\lfloor P \rfloor$, the largest integer less than or equal to P. [197]

3. Solve for p if $A^p = 2$, and

$$A = \prod_{k=0}^{\infty} e^{k \cdot (\ln 2)^k}.$$

$\left[(1 - \ln 2)^2\right]$

4. Thor, Captain America and Spiderman split 2001 candies among themselves. Due to seniority perks, Thor must get strictly more candies than Captain America, who must get strictly more candies than Spiderman, who must get at least one candy. How many ways can this be done? [332 667]

5. Give all integer values of x in the inclusive range $[1, 2018]$ such that $x^4 + 1020x^3 - 4x^2 + 2039x + 1$ is divisible by 1019, a prime number. $[1, 492, 524, 1010, 1511, 1543]$

Very difficult, eight points each

1. Let $s_0 = 6, s_1 = 6$, and $s_n = 2s_{n-1} + 8s_{n-2}$ for $n \geq 2$. Define

$$A_n = \sum_{i=0}^{n} s_i.$$

Find A_{2018}. Express your answer in the form $a^b + c^d$, where a, b, c, and d are positive integers. $\left[4^{2019} + 2^{2019}\right]$

2. Let $\{a_i\}$ be an infinite sequence of integers such that for $n \geq 1$,

$$a_{n+2} = 7a_{n+1}^2 + a_n$$

where $a_1 = 1$ and $a_2 = 25$. What is the remainder when $a_{2018!}$ is divided by 41? [18]

3. Thor and Loki were directed by Odin to connect 2018 realms by Rainbow Bridges. Initially, the realms are completely isolated from each other. Thor and Loki create 2017 Rainbow Bridges, each of which will directly connect only two realms, and such that it should then be possible to travel between any two realms through a network of bridges. Afterwards, Odin will give each realm an official Asgardian banner, with each containing exactly one of the 20 Royal Symbols. Odin will do this in such a way that any two realms directly connected by a Rainbow Bridge will receive different Royal Symbols. He is then expected to count the number of ways N of assigning the realms to banners. If Thor had his way, he would assign the bridges so that N is as large as possible. Let's call this maximum value N_{max}. If Loki had his way, he would assign the bridges so that N is as small as possible. Let's call this minimum value N_{min}. Find $N_{max} + N_{min}$, expressed as a product of prime powers. $[2^3 \cdot 5 \cdot 19^{2017}]$

Semifinals A

Easy, two points, thirty seconds

1. Find the number of factors[1] of $6^{10} + 2 \cdot 6^{12}$. [242]

2. Dr Strange has 6 001 unique books in the New York Sanctum's Library. How many ways can he choose any 5 998 of them to send to Hong Kong? [35 999 999 000]

3. Vision has N cookies and wants to distribute cookies equally, each undivided, to his friends. If he wanted to give 4 cookies to each friend, he can give cookies to at most 19 friends. If he wanted to give 5 cookies to each friend, he can give cookies to at most 15 friends. If he wanted to give 7 cookies to each friend, he can give cookies to at most 10 friends. How many cookies does he have? [76]

4. An ant rests somewhere on the circumference of the bottom base of a right circular cylinder with radius $\frac{7}{\pi}$ m. There is a sugar cube it wants to reach, somewhere along the circumference along the surface of the cylinder. The ant can climb up along the surface of the cylinder. If it goes westward, then the shortest path to the sugar is 13 m long. If it goes eastward, then the shortest path to the sugar is 15 m long. How tall (in meters) is the cylinder? [12 m]

5. Find the sum of $1 + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \cdots$. [4]

Average, three points, sixty seconds

1. Find the area of the equilateral triangle inscribed in the circle: $x^2 + y^2 - 9x + y + 4.5 = 0$. $[12\sqrt{3}]$

2. Simplify: $\log_4 19^{\log_{19} 4^{\log_4 19^{\log_{19} 4^{\log_4 19^{\log_{19} 16}}}}}$. [2]

3. It is believed that 1 in 10 000 humans are bitten by a radioactive spider (unbeknownst to them) and are given superpowers! Suppose you get tested to see if you have spider-superpowers or not. Just like with any other test, it may give a false result – it may say you have powers when you actually don't, or say you don't have powers when you actually do. The test is correct 99% of the time when you have a superpower, and also correct 99% of the time when

[1] It was not clarified that they were looking for positive factors; hence 484 was also accepted.

you don't have a superpower. If your test says you have spider superpowers, what is the probability that you actually have powers? $\left[\dfrac{1}{102}\right]$

4. Give all real solutions (x, y) to the following system:

$$4^{xy-3} = 2^{3x-4y}$$
$$\sqrt{2.5+y} + \sqrt{11+x} = 7.$$

$$\left[\left(-2, \frac{27}{2}\right), \left(14, \frac{3}{2}\right)\right]$$

5. Let S be the set of three-digit integers whose digits are from the set $\{1, 2, 3, 4, 5\}$. Three distinct numbers are chosen from S. What is the probability that no two of these three numbers have the same hundreds, tens, or units digit? $\left[\dfrac{144}{1271}\right]$

Difficult, five points, ninety seconds

1. In $\triangle ABC$, D and E lie on sides CA and AB such that $BE = 6$ and $CD = 10$. Let M and N be the midpoints of segments BD and CE, respectively. If $MN = 7$, then what is the measure of $\angle BAC$? [120°]

2. What is the remainder when $20\,182\,018^{1\,111\,111\,999\,999}$ is divided by $10^7 + 19$ (a prime number)? [181 980]

3. Find all ordered pairs (x, y) such that x and y satisfy the set of equations:

$$x^2 + xy = 12$$
$$4xy + 6y^2 = -10$$

$$\left[(4, -1), (-4, 1), \left(-3\sqrt{3}, \frac{5\sqrt{3}}{3}\right), \left(3\sqrt{3}, -\frac{5\sqrt{3}}{3}\right)\right]$$

4. Evaluate: $\displaystyle\sum_{k=1}^{2018} \arcsin\left(\frac{1009-k}{2018}\right)$. Recall that $\arcsin(x)$ returns a value in the range $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ radians. Give your answer in radians. $\left[-\dfrac{\pi}{6}\right]$

5. Find the number of trailing zeros of the expression $\displaystyle\prod_{i=1}^{1009} \frac{(2i)!}{(2i-1)!}$. [250]

Semifinals B

Easy, two points, thirty seconds

1. Find x and y if

$$4x^2 + y^2 + 1 + 4xy + 4x + 2y + 225x^2 - 30x + 49y^2 - 14y + 210xy + 1 = 0.$$

[(8, −17)]

2. Thanos, Thor, Hulk, Hela, and Hank are standing in a straight line in some random order. In how many ways can they be arranged such that if I read the initials of their names from left to right, the initials will NOT form a palindrome (i.e. the same when read forwards or backwards)? [96]

3. Solve for n in the following: $\ln\left(1 + e^{2018}\right) = 2n + \ln\left(1 + e^{-2018}\right)$. [1009]

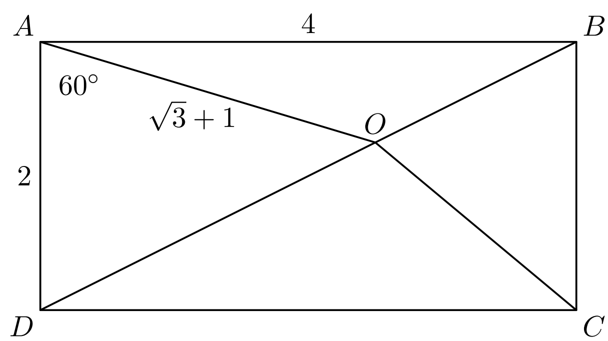
4. Consider the triangle ABC such that $AB = BC = 1$ and $\angle ABC$ is right. If point D is chosen on side AC such that it divides AC into two segments whose lengths are in the ratio $1 : 2$, find the length of BD. $\left[\dfrac{\sqrt{5}}{3}\right]$

5. A triangle ABC is constructed such that $AB = 3$, $BC = 4$, $AC = 5$. Another triangle ACD is constructed such that $AD = 12$, $CD = 13$, and AC intersects with BD. What is the distance between point D and line BC? $\left[\dfrac{63}{5}\right]$

Average, three points, sixty seconds

1. Let $A = \log_3 2^{\log_4 3^{\log_5 4^{\cdots \log_{2017} 2016^{\log_{2018} 2017}}}}$. Find 2018^A. [2]

2. Given rectangle $ABCD$ shown below, find the value of OC^2.



$[10 - 4\sqrt{3}]$

3. How many possible ordered pairs (a, b) are there such that a and b are positive integers, $\mathrm{GCD}(a, b) = 300$, and $\mathrm{LCM}(a, b) = 90\,000$? [8]

4. Let A be the smallest positive integer such that $\log_{10}\log_{10} A \geq 2$. How many positive divisors does A have? [10 201]

5. How many ways can a 1×20 board be completely tiled with exactly 16 game pieces, where a game piece is either a 1×1 square or a 1×2 domino? No part of each game piece should go outside the board, game pieces should not overlap, and no part of the board should be left uncovered. Except for their sizes, the game pieces are indistinguishable. [1820]

Difficult, five points, ninety seconds

1. In $\triangle ABC$, let M be the midpoint of CA, and Y be on the segment AB where $AY = 4$ and $BY = 6$. Suppose X lies on the segment CY such that $\angle ABX = \angle CXM$, and that $XY = 3$. Find the length of CY. [14]

2. Suppose we have positive integers m, n, where n is odd, that satisfy $\frac{1}{m} + \frac{4}{n} = \frac{1}{12}$. Find the maximum possible value of $m + n$. [637]

3. Let's play a game with Black Widow! She has an integer N, whose value is initially 2. The following is a move: she chooses a random integer k in the range $[1, 1000]$ and replaces N with N^k. Black Widow wins if after exactly 2018 moves have been performed, the units digit of N is 2. Find the probability that she wins the game. Express your answer as a fraction with products of prime powers. $\left[\frac{1}{2^{2019}}\right]$

4. Find the remainder when $\left(10x^{10} - 9x^9 + \cdots + 2x^2 - x\right)$ is divided by $\left(x^2 + x - 2\right)$. $[-6143x + 6148]$

5. Peter Parker forgot to study for his History of Superheroes test! fortunately, it's only a 6-item multiple choice test. For each item, Peter can choose a, b, or c. Peter is going to answer such that each letter is chosen at least once. How many ways can he answer his test? [540]

Finals

Teams sequentially choose question order, and can skip between rounds. For mechanics, see the Sipnayan 2018 JHS file at `https://cjquines.com/files/sipnayan2018jhs.pdf`.

The question order this year is E-RL, A-RL, D-RL, E-SL, V-SL, V-RL, A-SL, D-SL, A-TM, V-TM, D-MN, V-PW, E-PW, A-MN, D-TM, V-MN, E-MN, E-TM, A-PW, D-PW.

Easy, 2 points, 30 seconds

E-MN. Dr Strange flips a fair coin, rolls a fair six-sixded die, and grabs a random letter of the alphabet. What is the probability that at least one of the following is satisfied: his coin leads Heads; his die lands on a 6; his letter is a vowel? $\left[\frac{69}{104}\right]$

E-PW. How many numbers less than 2018 can be expressed as the sum of three consecutive positive odd numbers? [335]

E-TM. In the sequence: $\sqrt{1}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \cdots, \sqrt{10009}, \sqrt{10011}$, how many are rational? [50]

E-RL. A root of $ax^2 - 10x + 3 = 0$ is two-thirds the other root. Find the sum of the roots. $\left[\frac{5}{4}\right]$

E-SL. Let $A(x)$ be a degree 4 polynomial with zeroes at $x = -6, -4, 2$, and 3 with a leading coefficient of 1. Find the sum of all the coefficients of $(A(x))^2$ when it is expanded. The coefficient of x^0 should be included in the sum. [4 900]

Average, 3 points, 60 seconds

A-MN. Suppose positive integers a and b satisfy $\frac{7}{8} < \frac{a}{b} < \frac{8}{9}$. Find $\frac{a}{b}$ such that b is as small as possible. $\left[\frac{15}{17}\right]$

A-PW. Solve the system of equations

$$\begin{aligned} xyz &= -1 \\ \frac{1}{xy} + \frac{1}{yz} + \frac{1}{xz} &= -\frac{1}{6} \\ \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= \frac{11}{6}. \end{aligned}$$

$\left[\left(-\frac{3}{2}, \frac{2}{3}, 1\right) \text{ and permutations}\right]$

A-TM. Given the system

$$\begin{aligned} (x+y)\left(x^2 - xy + y^2\right) &= 13832 \\ xy(x+y) &= 13680, \end{aligned}$$

find all ordered pairs (x, y) such that $x < y$. $[(18, 20)]$

A-RL. Find the value of $\left\lfloor \frac{2018^5}{2016 \cdot 2017^3} - \frac{2016^5}{2018 \cdot 2017^3} \right\rfloor$. [12]

A-SL. Find the sum of the series $1 + (1+2) + (1+2+3) + \cdots + (1+2+3+\cdots+50)$. [22 100]

Difficult, 5 points, 90 seconds

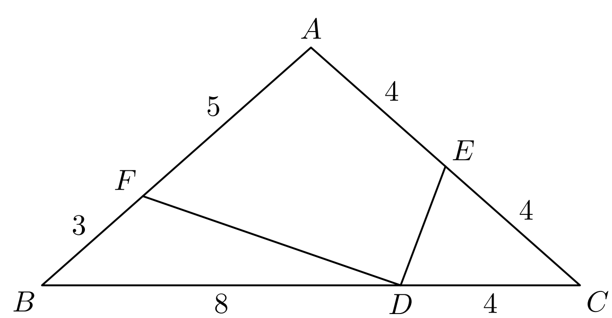
D-MN. Real numbers a and b satisfy $a^3 + b^3 + 3ab = 1$. Find the minimum possible value of $a + b$. $[-2]$

D-PW. Evaluate $\sum_{k=1}^{2018} \sin(15k^\circ)$. $\left[\frac{\sqrt{6} - \sqrt{2} + 2}{4}\right]$

D-TM. A Pym Particle, represented by a point, rests in one corner of a square container, 1 meter long on each side. The particle then begins to travel at a constant velocity of 1 m/s at an angle of $\arctan\left(\frac{9}{16}\right)$. If it hits an edge of the square, it reflects off the side such that the angle of incidence is equal to the angle of reflection. If it hits a corner of the square, it vanishes. How far (in meters) does the particle get to travel before it vanishes? $[\sqrt{337}]$

D-RL. A regular polygon is rotated 25.5° about its center and becomes coincident with its original position. Find the minimum number of sides of this polygon. [240]

D-SL. In the figure below, $AF = 5$, $FB = 3$, $BD = 8$, $DC = 4$, $CE = 4$, and $EA = 4$, find $FD^2 : DE^2$.



$[37 : 8]$

Very Difficult, 8 points, 120 seconds

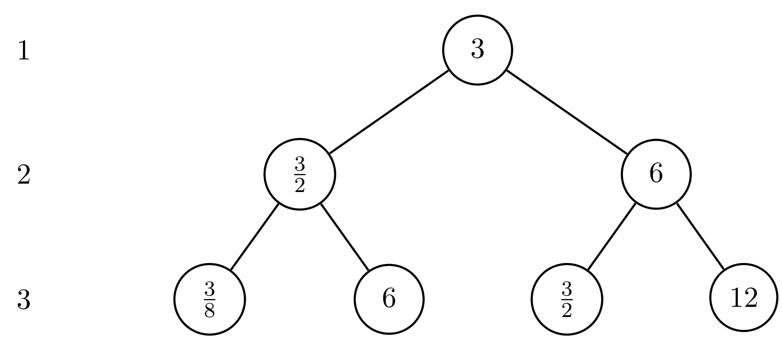
V-MN. Suppose Thanos keeps rolling a fair six-sided die until he rolls three composite numbers in a row. On average, how many times will he have to roll die up until the time this happens? [39]

V-PW. Let $ABCD$ be a quadrilateral such that AB and CD have lengths 15 and 27, respectively. Suppose X_1 and X_2 lie on the side DA such that $AX_1 = X_1X_2 = X_2D$ and that Y_1 and Y_2 lie on the side BC such that $BY_1 = Y_1Y_2 = Y_2C$. If X_1Y_1 has length 16, then what is the length of X_2Y_2?[2] $\left[2\sqrt{106}\right]$

V-TM. Thanos has a set S_1 of fractions: $\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{2018}\right\}$. He uses the Reality Stone and instantaneously generates another set S_2, whose elements are every non-empty subset of S_1. Thor picks a random element from S_2 and, using Stormbreaker, takes the product of all the elements in that subset. If, because of the Time Stone, he is forced to do this infinitely many times, what should he expect his average result to tend towards? $\left[\dfrac{2017}{2\left(2^{2017}-1\right)}\right]$

V-RL. Peter Parker's tutorial class has 5 seats lined up in a column – that is from front to back. The teacher then decides to rank each of the students by their performance in class, such that there are no ties. A student will be happy if the person sitting in the seat directly in front of him or her has a lower class rank. The person at the very front is never happy. How many ways can the students take their seats such that there are exactly 2 happy students? [66]

V-SL. Black Panther is mathematically decorating Wakanda's Christmas Tree which will have 2018 levels of ornaments, with Level 1 on top, Level 2 just below it, all the way down to Level 2018. On Level 1 is a single ornament. Every ornament is connected to exactly two ornaments in the level below it. Except for the ornament in Level 1, every ornament is connected to exactly one ornament in the level above it. Each ornament is numbered according to the following rules: (a) The ornament on level 1 has the number 3. (b) If X is the number on an ornament in Level $(n-1)$, the two ornaments below it are numbered $\dfrac{X}{2^{2^{n-2}}}$ and $X\left(2^{2^{n-2}}\right)$. What is the sum of the values of all ornaments on Level 2018?



$\left[\dfrac{2^{2^{2018}}-1}{2^{2^{2017}}-1}\right]$

[2] Apparently, the category of this question was "geometry question written by Kyle Dulay".