Semifinal Round A

Easy: 2pts, 30s

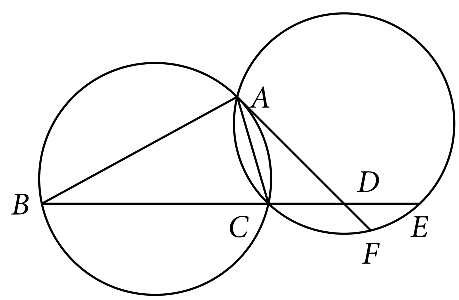
SFA E-1 A tank is filled with 30 L of water. Buzz's drain removes water from the tank at a fixed rate of 4 mL/s, while Woody's drain removes water from the tank at a fixed rate of 5 mL/s. At 12:00 PM, Woody starts draining water from the tank. At what time should Buzz start draining water from the tank so that together, they finish emptying the tank at exactly 1:00 PM? [12 : 10 PM]

SFA E-2 Toys last a long time. Woody is currently 24 years old while Jessie is 20 years old. How old will Jessie be when she is $\frac{2018}{2019}$ of Woody's age? [8072]

SFA E-3 What is the last digit of $(2019 - 51)^{2019+51}$ when it is evaluated? [4]

SFA E-4 Gabby Gabby orders six of her puppers to each independently roll a fair six-sided die. What is the chance that in the results, each of the numbers from 1 to 6 appears exactly once? $\left[\frac{5}{324}\right]$

SFA E-5 Let ω be the cirucumcircle of triangle ABC. Let the tangent to ω at A meet the line BC at point D such that C lies between B and D. Construct E such that D is the midpt of segment CE. Let AD meet the circumcircle of ACE again at point F. If $BC = 3$, $CE = 2$, then what is the length of DF? $\left[\frac{1}{2}\right]$



Average: 3pts, 60s

SFA A-1 Define $\lfloor x \rfloor$ as the greatest integer less than or equal to x. How many distinct numbers are there among $\left\lfloor \frac{1^3}{100} \right\rfloor, \left\lfloor \frac{2^3}{100} \right\rfloor, \left\lfloor \frac{3^3}{100} \right\rfloor, \ldots, \left\lfloor \frac{100^3}{100} \right\rfloor$? [97]

SFA A-2 A power of 5 is a positive integer whose only prime factor is 5. What is the largest power of 5 that divides $1 \times 3 \times 5 \times 7 \times \cdots \times 2019$? $[5^{252}]$

SFA A-3 Find the last digit of the sum $(1^{2019} + 2^{2019} + 3^{2019} + 4^{2019} + \cdots + 2018^{2019}) + 2019^{2018}$. All the exponents are 2019 except for the last term. [2]

SFA A-4 Suppose that function $f\left(\frac{a+b+c}{4}\right) = \frac{f(a)+f(b)+f(c)}{3}$ for all real numbers a, b, and c. Find all possible values of $f(2019) - f(673) + 11$. [11]

SFA A-5 Find all triples (x, y, z) of real numbers such that $3x^2 + 2y^2 + 2z^2 + 5 = 2z + 4y + 2z + 2yz$. $\left[\left\{\left(\frac{1}{3}, \frac{5}{3}, \frac{4}{3}\right)\right\}\right]$

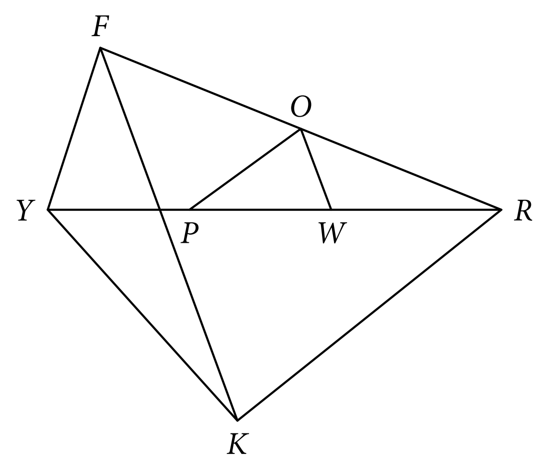
Difficult: 5pts, 90s

SFA D-1 Let S be the first 100 primes. Call a prime p "antique" if p^2 is one more than a multiple of 24. What part of S is antique? [98%]

SFA D-2 Find the value of $\sqrt[16]{37\,634 - \cfrac{1}{37\,634 - \cfrac{1}{37\,634 - \cdots}}}$. Express your answer in the form $\dfrac{a\sqrt{b} + c\sqrt{d}}{e}$ where a, b, c, d, e are positive integers, b and d are square-free integers, and a, c, and e are relatively prime. $\left[\dfrac{\sqrt{6}+\sqrt{2}}{2}\right]$

SFA D-3 Let $P(x)$ be a non-constant polynomial such that, if r is a real number such that $P(r) = 0$, then $P(r^2 - 3r + 4) = 0$. At most how many distinct roots can $P(x)$ have? [2]

SFA D-4 The diagram shows the quadrilateral $FRKY$, where W is a point on YR such that $FY = WR$. The points O and P are midpoints of FR and YW respectively. If $WR = 3$, $YK = 5$, $FK = 7$, and $(4OP - FY)^2 = 45$, find the measure of $\angle RYK$. $\left[48^\circ \text{ or } \dfrac{4\pi}{15}\right]$



SFA D-5 Find $\gcd(4^{51} + 1, 4^{2019} + 1)$. [65]

Semifinal Round B

Easy: 2pts, 30s

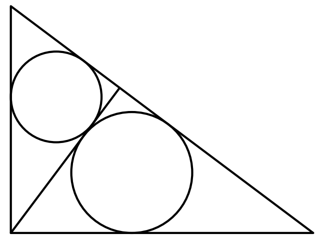
SFB E-1 If all sides of a square have their lengths increased by 2 meters, the area of the square doubles. What is the original side length of the square in meters? $\left[(2 + 2\sqrt{2})\text{ m}\right]$

SFB E-2 In Rex's video game, a random integer n in the range $0 \le n \le 100$ is generated. If $n^2 \le 100$, then his character performs a critical hit. What is the chance that the next two hits are critical hits? $\left[\dfrac{121}{10\,201}\right]$

SFB E-3 Determine the value of $\sin 75^\circ + \cos 75^\circ$. $\left[\dfrac{\sqrt{6}}{2}\right]$

SFB E-4 What are the last two digits of $3^{4^{5^6}}$? [81]

SFB E-5 In the right triangle below, an altitude is drawn from the right angle to the hypotenuse. Circles are inscribed within each of the smaller triangles. What is the distance between the centers of these circles? $[5\sqrt{2}]$



Average: 3pts, 60s

SFB A-1 Given that $x^2+y^2+z^2=10$ for some real numbers x, y, z, find the maximum value of $xy+xz-yz$. [5]

SFB A-2 Let P be a point inside triangle ABC such that $\angle ACP = \angle BCP = 10°$. Line AP intersects BC at a point D such that $\angle ADC = 135°$ and $\angle ABC = 110°$. Find the measure of $\angle CBP$. [55°]

SFB A-3 Woody has a total of 18 identical match sticks. Using the match sticks are sides, he formed a regular 18-gon. With the definition that an isosceles triangle has at least two equal sides, how many non-isosceles triangles with vertices found among the vertics of the regular 18-gon are there? [684]

SFB A-4 Find the polynomial $P(x)$ of the lowest possible degree such that the graph of $y=P(x)$ passes through the points $(2,0)$, $(1,9)$, and $(5,1)$. $\left[\frac{7}{3}x^2-16x+\frac{68}{3}\right]$

SFB A-5 Let $f:\mathbb{N}\to\mathbb{N}$ be a strictly increasing function satisfying $f(f(f(n)))=4n$ for all natural numbers n. Find the value of $f(69)$. [133]

Difficult: 5pts, 90s

SFB D-1 You are given that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$. Evaluate $\sum_{n=2}^{\infty}\frac{1}{n^4-2n^2+1}$. $\left[\frac{\pi^2}{12}-\frac{11}{16}\right]$

SFB D-2 Gabby Gabby has a 30×30 chessboard and two identical bishops. Recall that in one move, a bishop can jump any number of steps in a "diagonal" 45° angle, and attack any chess piece in its way. How many ways can she place the two bishops on two different squares on the chessboard so that they can attack each other? [17 110]

SFB D-3 Suppose that a, b, c are positive integers such that $(a^2+b^2+c^2)^2-2(a^4+b^4+c^4)=63$. Find all possible values of $a+b+c$. [7, 9]

SFB D-4 Find the last two digits of $1\times 2017!+2\times 2016!+3\times 2015!+\cdots+2016\times 2!+2017\times 1!$. [35]

SFB D-5 The circumcircle of non-obtuse triangle TOY has center A. Point B is on ray YO past O. If $BT=15$, $OY=16$, and $OB=9$, find the area of the polygon $TAYO$ if the circumcircle has diameter 20. [96]

Final Round

Easy: 2pts, 30s

F E-Woody How many ordered pairs of positive integers m and n are there such that $\frac{1}{m}+\frac{1}{n}=\frac{1}{143}$? [9]

F E-Buzz Find the ratio between the length of the inradius and of the circumradius of a triangle with sides 20, 21, and 29. [12 : 29]

F E-Hamm Let A, B, and C be points initially at $(0,5)$, $(1,0)$, and $(6,1)$ respectively. Suppose that A, B, and C are moving down, right, and up at 1 unit per second, respectively. After how many seconds will the area of triangle ABC achieve its minimum value? [2]

F E-Potato Find the value of the sum $\log_{1\times 2019} 1 + \log_{2\times 2018} 2 + \log_{3\times 2017} 3 + \cdots + \log_{2019\times 1} 2019$. $\left[\frac{2019}{2}\right]$

F E-Forky How many positive integers less than 2310 are relatively prime to it? [480]

Average: 3pts, 60s

F A-Woody Evaluate $\sqrt{\left(\frac{3-\sqrt{5}}{2}\right)+\sqrt{\left(\frac{3-\sqrt{5}}{2}\right)^2+\sqrt{\left(\frac{3-\sqrt{5}}{2}\right)^4+\sqrt{\left(\frac{3-\sqrt{5}}{2}\right)^8+\sqrt{\cdots}}}}}$. [1]

F A-Buzz Let $f(n)$ be the remainder when $(3^n-1)^n+(3^n+1)^n$ is divided by 9^n. How many positive divisors does $f(2019)$ have? [8084]

F A-Hamm Let a, b, and c be positive real numbers such that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=4$ and $\frac{b}{a}+\frac{c}{b}+\frac{a}{c}=5$. Find the value of $a^2\left(\frac{1}{b^2}-\frac{1}{c^2}\right)+\left(\frac{1}{c^2}-\frac{1}{a^2}\right)+c^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)$. [−11]

F A-Potato Let R be the remainder when $6^{83}+8^{83}$ is divided by 49. Find the product of the digits of R. [15]

F A-Forky The operation $*$ is defined as $a*b=(\log_b a^b)+ab$. Find all integers x such that $8*x=22$. [2]

Difficult: 5pts, 90s

F D-Woody What is the remainder when $(100!)^{100}\left(\left(\frac{1}{2}\right)^{99}+\left(\frac{2}{3}\right)^{99}+\left(\frac{3}{4}\right)^{99}+\cdots+\left(\frac{99}{100}\right)^{99}\right)$ is divided by 101? [100]

F D-Buzz Let S be the collection of all possible subsets of $\{1,2,3,\ldots,4000\}$. Then, if A is a set of integers, the function $f(A)$ returns the sum of all the elements of A, with $f(\{\})=0$. What is the average value of f over all elements of S? [4 001 000]

F D-Hamm Find the maximum value of k such that for all positive numbers a and b, $\frac{a^4}{b^2}+\frac{b^2}{a^4}\geq 2ab+k(a-b)^2$. [9]

F D-Potato Let P be a poinit inside unit square $ABCD$. What is the minimum value of $PC+PD+\sqrt{2}PA$? [$\sqrt{5}$]

F D-Forky Let $d_1, d_2, \ldots, d_n$ be the positive factors of 2019^3 in increasing order. Find the value of $\frac{2019^2}{d_1^2+2019^3}+\frac{2019^2}{d_2^2+2019^3}+\cdots+\frac{2019^2}{d_n^2+2019^3}$. $\left[\frac{8}{2019}\right]$

Very Difficult: 8pts, 120s

F VD-Woody Let $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$ be real numbers that satisfy the equations $\frac{(\sec\alpha+1)(\sec\beta+1)}{\sec\alpha\sec\beta}=\frac{7}{6}$ and $\frac{(\csc\alpha+1)(\csc\beta+1)}{\csc\alpha\csc\beta}=\frac{27}{14}$. Find the value of $\sin(\alpha+\beta)$. $\left[\frac{19}{21}\right]$

F VD-Buzz Define the Sipnayan-Fibonacci numbers by $F_1 = F_2 = 1$, and $F_n = 2019F_{n-1} + 1859F_{n-2}$ for $n \geq 3$. How many positive integers N are there such that N contains exactly 2019 digits, none of those digits are 0, and F_N is an even number? Express your answer as a product of prime factors. [3^{4037}]

F VD-Hamm In triangle ABC, E is a point on AB, D is a point on AC, and DB and CE intersect at point F. If $\angle ACE = 69°$, $\angle BCE = 42°$, $AD = DB$, and $\frac{BC}{CF} = \frac{BE}{EF}$, find the measure of $\angle ADE$. [113°]

F VD-Potato Given that x and y are real numbers such that $x > y > 0$, find the minimum value of $81x^3 + \frac{4}{xy - y^2}$. [60]

F VD-Forky Let $f(m)$ be the smallest integer value of n which maximizes the value of $\sum_{k=1}^{m-n+1} \binom{m-k}{n-1}$ where the binomial coefficient $\binom{a}{b}$ is the number of ways to form a committee of size b from a class of a students, if order doesn't matter. By convention $\binom{a}{b} = 0$ if $b < 0$ or $b > a$. Evaluate the sum $\sum_{m=1}^{2019} f(m)$. [1 019 090]