

Spelling out numbers

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1. Warm-up (NIMO April 2017, Michael Tang) Find $f(4\,000)^2 + f(40\,000)^2 + f(400\,000)^2$ where $f : \mathbb{Z} \rightarrow \mathbb{Z}$ is a function satisfying the following conditions:

- a) For every positive integer x , we have $f(-x) = f(x) + 8$.
- b) For $x \in \{3, 4, \dots, 9\}$, we have

$$f(x + 10) - f(10x) = 2 - \frac{1}{120}(x - 3)(x - 5)(x - 6)(x - 7)(x - 8)(x - 9).$$

- c) $f(10^{12}) \in \{10, 11\}$.
- d) For all $x \in \mathbb{Z}$ the sequence

$$\{f^n(x)\}_{n \geq 0} = x, f(x), f(f(x)), f(f(f(x))), \dots$$

eventually becomes constant at 4.

2. On arranging alphabetically:

- a) We sort the integers from 1 to 10^{10} alphabetically. What number appears first? What number appears last?
- b) We spell out the integers from 1 to 99 to form a list. We then arrange this list alphabetically. Which numbers don't change their places?
- c) We call the above the *alphabetical permutation* of a list. For which n do the integers from 1 to n have a fixed point in their alphabetical permutation?

3. Not having letters:

- a) What's so special about this sequence: 2, 4, 6, 30, 32, 34, 36, ...?
- b) The above sequence is called the *eban* sequence, because it consists of all the numbers without the letter 'e' when spelled out. Show that all eban numbers are even.
- c) Similarly, the *oban* numbers are those without the letter 'o'. What's the largest oban number? How many oban numbers are there?
- d) We can define *aban*, *iban* and *uban* numbers similarly. Of the numbers from 1 to 1000, which appears most often: aban, eban, iban, oban or uban numbers?
- e) Here is the beginning of the list of natural numbers without the digit 4: 1, 2, 3, 5, 6, 7, 8, 9, ... What's the difference between this list and the list of uban numbers?
- f) What's the largest iban number? How many iban numbers are there?

4. Having letters:

- a) What's the smallest number that contains all the vowels? What about all the vowels and the letter 'y'?
- b) What's the largest number with no repeated letters?
- c) What's the largest number which contains every vowel exactly once?
- d) Is the consonant to vowel ratio of a number unbounded? If not, which number achieves the maximum?